TUTORATO 11/05.

EX. 5.

$F: \mathbb{R}^3 \to \mathbb{R}^3$

$f(x_1, x_2, x_3) = (x_1 + x_3,\ 2x_2,\ x_1 + x_3)$

(a)

$$A = M_{\mathcal{E} \to \mathcal{E}}(F) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(b) $\ker f = ?$

$\ker f = \ker A$

$$\ker f : \begin{cases} x_1 + x_3 = 0 \longrightarrow x_3 = -x_1 \\ 2x_2 = 0 \longrightarrow x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

$\dim(\ker f) = 1 \qquad \text{BASE}(\ker f) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$

(c) f NON è invertibile perché

$\dim(\ker f) \neq 0.$

(d) $p_A(\lambda) = \det(\lambda I - A) =$

$$= \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ -1 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-1 \end{vmatrix} - \begin{vmatrix} 0 & \lambda-2 \\ -1 & 0 \end{vmatrix} =$$

$$= (\lambda-1)^2(\lambda-2) - (\lambda-2)$$

$$\overset{!}{=} (\lambda-2)(\lambda^2-2\lambda \cancel{+1} \cancel{-1})$$

$$\overset{!}{=} \lambda(\lambda-2)^2.$$

<u>AUTOVALORI:</u>

$\lambda_1 = 0 \quad \mu_1 = 1 \quad g_1 = 1$

$\lambda_2 = 2 \quad \mu_2 = 2 \quad g_2 = 2\,?$

(e) <u>AUTOSPAZI:</u>

· $A_{\lambda_1} = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

$\mathrm{Ker}(A-2I) = \,?$

$$A-2I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\mathrm{Ker}(A-2I): \begin{cases} -x_1 + x_3 = 0 \\ 0 = 0 \\ x_1 - x_3 = 0 \end{cases} \Longrightarrow \begin{cases} x_3 = x_1 \\ x_2 \text{ qls} \end{cases}$$

$$A_{\lambda_2} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

$$\Rightarrow g_2 = 2$$

$\mathrm{V})$ $\Rightarrow$ F DIAGONALIZZABILE

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

BASE DI AUTOVETTORI:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Ex.

$h \in \mathbb{R}$

$$A_h = \begin{pmatrix} 0 & 1 & 0 \\ h & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a)

$$\det(A_h - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ h & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} =$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} h & 1 \\ 0 & -\lambda \end{vmatrix} =$$

$$= -\lambda(\lambda^2 - 1) + h\lambda =$$

$$= -\lambda(\lambda^2 - 1 - h)$$

- $\lambda = 0$
- $\lambda^2 - 1 - h = 0 \longrightarrow \lambda^2 = 1 + h$

<u>I CASO)</u> $1 + h > 0 \iff \boxed{h > -1}$

$\lambda_1 = 0$
$\lambda_2 = \sqrt{1+h}$ $\Longrightarrow$ DIAGONALIZZABILE.
$\lambda_3 = -\sqrt{1+h}$

<u>II CASO)</u> $1 + h < 0 \iff \boxed{h < -1}$

$\longrightarrow$ 2 autovalori complessi coniugati

$\Longrightarrow$ NON DIAG. SU $\mathbb{R}$.

II CASO) $\boxed{h=-1}$

$\lambda = 0 \quad \mu = 3 \longrightarrow g = ?$

$$A_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

AUTOSPAZIO di $\lambda = 0$

$\dim A_0 = 1$

$$\begin{cases} y = 0 \\ -x + z = 0 \longrightarrow z = x \end{cases}$$

$$A_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

- $\boxed{h=-1} \Longrightarrow$ NON DIAG.

(b) $\bar{h}$ per cui $A_{\bar{h}}$ ha autovalori di $\mu > 1$.

$\longrightarrow \bar{h} = -1$

$$A_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$ AUTOSPAZIO $\lambda = 0$ $\bar{h} = -1$.

(c) $\boxed{h=3}$

$?P$ t.c. $P^{-1} A_3 P = D$

$h = 3 \Longrightarrow h > -1 \Longrightarrow$ DIAG. (CASO I)

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 2 \\ \lambda_3 = -2 \end{array}$$

- $\lambda_1 = 0$

$$\begin{cases} y = 0 \\ 3x + z = 0 \longrightarrow z = -3x \end{cases} \qquad A_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \right\rangle$$

- $\lambda_2 = 2$

$$\mathrm{Ker}(A_3 - 2I) = \mathrm{Ker} \begin{pmatrix} -2 & 1 & 0 \\ 3 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\begin{cases} -2x + y = 0 \longrightarrow y = 2x \\ 3x - 2y + z = 0 \quad 3x - 4x + z = 0 \\ y - 2z = 0 \qquad \hookrightarrow z = x \end{cases}$$

$$A_2 = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

- $\lambda_3 = -2$

$$\mathrm{Ker}(A_3 + 2I) = \mathrm{Ker} \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{cases} 2x + y = 0 \longrightarrow y = -2x \\ 3x + 2y + z = 0 \quad 3x - 4x + z = 0 \longrightarrow z = x \\ y + 2z = 0 \end{cases}$$

$$A_{-2} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ -3 & -1 & 1 \end{pmatrix} \longrightarrow D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(d) $\bar{h} = -1$

$(A_{\bar{h}})^3 = 0$

$A_{\bar{h}}$ simile a $\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=B}$?

NON SONO SIMILI perché $rk(B) = 1 \neq 2 = rk(A_{-1})$.

$$A_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(A_{-1})^2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$(A_{-1})^3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Ex.

$$A = \begin{pmatrix} 8 & -1 & 1 \\ 0 & t & 1 \\ -2 & 2 & 6 \end{pmatrix}$$

(a) ?t t.c.

$v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ sia AUTOVETTORE DI A.

$$Av = \lambda v, \quad \lambda \in \mathbb{R}$$

$$\begin{pmatrix} 8 & -1 & 1 \\ 0 & t & 1 \\ -2 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ t-1 \\ -6 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix} \Rightarrow \frac{\lambda = 6}{\boxed{t = 7}}$$

(b) $\boxed{t = 7}$

$$A = \begin{pmatrix} 8 & -1 & 1 \\ 0 & 7 & 1 \\ -2 & 2 & 6 \end{pmatrix}$$

$$p_A(\lambda) = \det(A - \lambda I) =$$

$$= \begin{vmatrix} 8-\lambda & -1 & 1 \\ 0 & 7-\lambda & 1 \\ -2 & 2 & 6-\lambda \end{vmatrix} =$$

$$= (8-\lambda)\begin{vmatrix} 7-\lambda & 1 \\ 2 & 6-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 7-\lambda & 1 \end{vmatrix} =$$

$$= (8-\lambda)\left[(7-\lambda)(6-\lambda)-2\right] - 2\left[-1-(7-\lambda)\right] =$$

$$= (8-\lambda)\left[4\underbrace{2-13\lambda+\lambda^2-2}_{\lambda^2-13\lambda+40}\right] + 2 + 14 - 2\lambda =$$

$$= -\lambda^3 + 13\lambda^2 - 40\lambda + 8\lambda^2 - 104\lambda + 320 + 16 - 2\lambda$$

$$= -\lambda^3 + 21\lambda^2 - 146\lambda + 336$$

$$\overset{!}{=} -(\lambda-6)(\lambda^2-15\lambda+56) \quad \to \lambda_1 = 6, \quad \to \lambda_{2,3}$$

$$\lambda^2 - 15\lambda + 56 = 0$$

$$\lambda_{2,3} = \frac{15 \pm \sqrt{225-224}}{2} = \begin{cases} 8 \\ 7 \end{cases}$$

$\lambda_1 = 6, \ \lambda_2 = 7, \ \lambda_3 = 8$

$\Longrightarrow$ A è DIAGONALIZZABILE.

- $\lambda_1 = 6 \longrightarrow v_1 = (1, 1, -1)$
- $\lambda_2 = 7 \longrightarrow v_2 = (1, 1, 0)$

$$\ker(A - 7I) = \ker \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

$$\begin{cases} x - y + z = 0 \\ z = 0 \\ -2x + 2y - z = 0 \end{cases} \longrightarrow \begin{cases} y = x \\ z = 0 \end{cases} \qquad A_7 = \left\langle \underset{\substack{\| \\ v_2}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \right\rangle$$

- $\lambda_3 = 8 \longrightarrow v_3 = (0, 1, 1)$

$$\ker(A - 8I) = \ker \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ -2 & 2 & -2 \end{pmatrix}$$

$$\begin{cases} z = y \\ -2x + \cancel{2y} - \cancel{2y} = 0 \end{cases} \longrightarrow \begin{cases} z = y \\ x = 0 \end{cases} \qquad A_8 = \left\langle \underset{\substack{\| \\ v_3}}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right\rangle$$

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow D = P^{-1}AP = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

(c) $A = \begin{pmatrix} 8 & -1 & 1 \\ 0 & 7 & 1 \\ -2 & 2 & 6 \end{pmatrix}$

? base ORTONORMALE di $\mathbb{R}^3$ formato da AUTOVETTORI di A?

$A \neq A^T \Rightarrow$ NON è possibile.

Ex.

$$A = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 3 & -1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & -3 \end{pmatrix}$$

(a)

$p_A(\lambda) = \det(A - \lambda I) =$

$$= \begin{vmatrix} -5-\lambda & 0 & 0 & 0 \\ 3 & -1-\lambda & 0 & 2 \\ 2 & 0 & 1-\lambda & 0 \\ -1 & 4 & 0 & -3-\lambda \end{vmatrix} =$$

$$= (-5-\lambda) \begin{vmatrix} -1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 4 & 0 & -3-\lambda \end{vmatrix} =$$

$$= (-5-\lambda)(1-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 4 & -3-\lambda \end{vmatrix} =$$

$$= (-5-\lambda)(1-\lambda)\Big[\underbrace{\underbrace{(-1-\lambda)(-3-\lambda)}_{\lambda^2+4\lambda+3} - 8}_{\lambda^2+4\lambda-5}\Big]$$

$$= (-5-\lambda)(1-\lambda)(\lambda^2+4\lambda-5)$$

$$\overset{!}{=} (-5-\lambda)(1-\lambda)(\lambda+5)(\lambda-1)$$

$\lambda_1 = 1$, $\mu_1 = 2$, $g_1 = 2$

$\lambda_2 = -5$, $\mu_2 = 2$, $g_2 = 1$ $\Rightarrow$ NON DIAG.

$\mu_2 \neq g_2$

- $\boxed{\lambda_1 = 1}$

$$\text{Ker}(A-I): \begin{pmatrix} -6 & 0 & 0 & 0 \\ 3 & -2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ -1 & 4 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 = 0 \\ x_4 = x_2 \end{cases}$$

x_2 qls

x_3 qls $\Rightarrow \dim = 2$ $\quad A_1 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

• $\boxed{\lambda_2 = -5}$

$$\mathrm{Ker}(A+5I): \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 2 \\ 2 & 0 & 6 & 0 \\ -1 & 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3x_1 + 4x_2 + 2x_4 = 0 \\ 2x_1 + 6x_3 = 0 \longrightarrow x_1 = -3x_3 = 0 \\ -x_1 + 4x_2 + 2x_4 = 0 \end{cases}$$

$$-9x_3 + 4x_2 + 2x_4 = 0 \longrightarrow 2x_4 = 9x_3 - 4x_2$$

$$3x_3 + 4x_2 + 2x_4 = 0$$

$$3x_3 + \cancel{4x_2} + 9x_3 - \cancel{4x_2} = 0.$$

$$12x_3 = 0 \longrightarrow x_3 = 0$$

$$\begin{cases} x_1 = 0 \\ x_3 = 0 \\ x_4 = -2x_2 \end{cases} \qquad A_{-5} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} \right\rangle$$

$$\Longrightarrow \dim A_{-5} = 1 \Longrightarrow g_2 = 1$$

$\Longrightarrow$ A NON DIAGONALIZZABILE.

(b) $U \subset \mathbb{R}^4$

$U = \langle u_1, u_2 \rangle$, $u_1 = (0, 1, 0, -1)$

$u_2 = (0, 3, 0, 1)$.

$\forall v \in U \Rightarrow Av \in U$.

$$A u_1 = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 3 & -1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 7 \end{pmatrix}$$

$$A u_2 = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 3 & -1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 9 \end{pmatrix}$$

$(0, -3, 0, 7) = \alpha_1 u_1 + \alpha_2 u_2$

$(0, -1, 0, 9) = \beta_1 u_1 + \beta_2 u_2$

$$\begin{cases} 0 = 0 \\ -3 = \alpha_1 + 3\alpha_2 \\ 0 = 0 \\ 7 = -\alpha_1 + \alpha_2 \end{cases}$$

$\rightarrow \alpha_1 = -3\alpha_2 - 3 = -6$

$+3\alpha_2 + 3 + \alpha_2 = 7$

$\hookrightarrow \alpha_2 = 1$

$$\Rightarrow \underline{A u_1 = -6 u_1 + u_2}$$

$$\begin{cases} 0 = 0 \\ -1 = \beta_1 + 3\beta_2 \\ 9 = -\beta_1 + \beta_2 \end{cases}$$

$\rightarrow \beta_1 = -3\beta_2 - 1 = -7$

$3\beta_2 + 1 + \beta_2 = 9$

$4\beta_2 = 8 \rightarrow \beta_2 = 2$

$$\Rightarrow A u_2 = -7 u_1 + 2 u_2 .$$

(c) $f : U \to U$

$f(v) = Av$ $\quad \boxed{\mathcal{U} = \{u_1, u_2\}}$

$B = M_{\mathcal{U} \to \mathcal{U}}(f) = ?$

$$B = \begin{pmatrix} -6 & -7 \\ 1 & 2 \end{pmatrix}.$$

$\underline{f(u_1) = Au_1}$ $\quad f(u_2) = Au_2$