TUTORATO 04/05

EX. 3

$$A_K = \begin{pmatrix} 1 & 2 & K \\ K-1 & 1 & 2 \\ 1 & -1 & 2-K \end{pmatrix}$$

(a) ?K NON iniettivo.

INIETTIVO
$\Updownarrow$
$\ker A_K = \{\underline{0}\} \Rightarrow rk\, A_K = 3$.

$$\det A_K = \begin{vmatrix} 1 & 2 & K \\ K-1 & 1 & 2 \\ 1 & -1 & 2-K \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 2 \\ -1 & 2-K \end{vmatrix} - 2 \begin{vmatrix} K-1 & 2 \\ 1 & 2-K \end{vmatrix} + K \begin{vmatrix} K-1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 2-K+2 - 2\left[(K-1)(2-K)-2\right] +$$

$$+ K\left[\overset{-K+\cancel{1}-\cancel{1}}{-(K-1)-1}\right] =$$

$$= 2 - k + 2 + 2\underbrace{(k-1)(k-2)}_{k^2-3k+2} + 4 - k^2 =$$

$$= k^2 - 7k + 12 = (k-3)(k-4).$$

$$\det A_k = 0 \iff \boxed{k=3} \vee \boxed{k=4}.$$

- $\boxed{k=3}$.

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\mathrm{Im}\, A_3 = \left\langle \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}} \right\rangle$$

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + y + 2z = 0 \\ x - y - z = 0 \end{cases} \longrightarrow \mathrm{Ker}\, A_3 = \left\langle \underbrace{\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}} \right\rangle.$$

- $\boxed{k=4}$

$$A_4 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathrm{Im}\, A_4 = \left\langle \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$\ker A_4 \to \begin{cases} x+2y+4z=0 \\ 3x+y+2z=0 \\ x-y-2z=0 \end{cases}$$

$$\Rightarrow \ker A_4 = \left\langle \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle.$$

(c) A_k INIETTIVO $\Leftrightarrow k \neq 3, 4$.

$$A_k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ INVERTIBILE $\Leftarrow \det A_k \neq 0$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_k^{-1} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$A_k = \begin{pmatrix} 1 & 2 & k \\ k-1 & 1 & 2 \\ 1 & -1 & 2-k \end{pmatrix} \quad \det A_k = (k-3)(k-4).$$

$$A_k^{-1} = \frac{1}{\det A_k} S_k^T$$

$$S_k = \begin{pmatrix} -(k-4) & (k+1)(k-4) & -k \\ k-4 & -2(k-1) & 3 \\ -(k-4) & (k+1)(k-2) & -2k+3 \end{pmatrix}$$

$$\delta_{11} = \begin{vmatrix} 1 & 2 \\ -1 & 2-k \end{vmatrix} = 2-k+2 = 4-k$$

$$\delta_{12} = -\begin{vmatrix} k-1 & 2 \\ 1 & 2-k \end{vmatrix} = -\left[\overset{-(k-1)(k-2)}{(k-1)(2-k)} - 2\right]$$

$$\overset{!}{=} +(k^2-3k+2)+2$$

$$\overset{!}{=} -[k^2+3k-4]$$

$$\overset{!}{=} k^2-3k+4 = (k+1)(k-4)$$

$$\delta_{13} = \begin{vmatrix} k-1 & 1 \\ 1 & -1 \end{vmatrix} = -k+\cancel{1}-\cancel{1} = -k$$

$$\delta_{21} = -\begin{vmatrix} 2 & k \\ -1 & 2-k \end{vmatrix} = -[4-2k+k]$$

$$\overset{!}{=} k-4$$

$$\delta_{22} = \begin{vmatrix} 1 & k \\ 1 & 2-k \end{vmatrix} = 2-k-k = -2k+2$$

$$\overset{!}{=} -2(k-1)$$

$$\delta_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -(-1-2) = 3$$

$$\delta_{31} = \begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = 4 - k = -(k-4)$$

$$\delta_{32} = -\begin{vmatrix} 1 & k \\ k-1 & 2 \end{vmatrix} = -(2 - k(k-1))$$

$$= -2 + k^2 - k$$

$$= k^2 - k - 2 = (k+1)(k-2)$$

$$\delta_{33} = \begin{vmatrix} 1 & 2 \\ k-1 & 1 \end{vmatrix} = 1 - 2k + 2$$

$$= -2k + 3$$

$$A_k^{-1} = \begin{pmatrix} -\frac{1}{k-3} & \frac{1}{k-3} & \frac{-1}{k-3} \\ \frac{k^2-3k+4}{(k-3)(k-4)} & \frac{-2(k-1)}{(k-3)(k-4)} & \frac{(k+1)(k-2)}{(k-3)(k-4)} \\ -\frac{k}{\det A_k} & \frac{3}{\det A_k} & \frac{-2k+3}{\det A_k} \end{pmatrix}$$

Ex. 4.

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

a) $$\det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \overset{?}{\neq} 0$$

$$\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 - 2(2+1)$$

$$= 2 - 6 = -4 \neq 0 .$$

$\Rightarrow \mathcal{A}$ è BASE di $\mathbb{R}^3$.

$$\det \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix} =$$

$$= 3 \begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} = 3(1+3) - 3$$

$$= 9 \neq 0 .$$

$\Rightarrow \mathcal{B}$ è BASE di $\mathbb{R}^3$.

(b) $M_{A \to B} = ?$

$$\to M_{A \to \mathcal{E}} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$M_{\mathcal{E} \to A} = (M_{A \to \mathcal{E}})^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} =$$

$$= \frac{1}{-4} S^T$$

$$\left(S = \begin{pmatrix} 2 & -3 & -1 \\ -4 & 2 & 2 \\ -2 & 1 & -1 \end{pmatrix} \right.$$

$$= -\frac{1}{4} \begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1 & 1/2 \\ 3/4 & -1/2 & -1/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}$$

$$\to M_{B \to \mathcal{E}} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix}$$

$$M_{\mathcal{E} \to B} = \left(M_{B \to \mathcal{E}} \right)^{-1} =$$

$$= \frac{1}{9} S^T \qquad S = \begin{pmatrix} 4 & 1 & 3 \\ -3 & -3 & -9 \\ -1 & 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4/9 & -1/3 & -1/9 \\ 1/9 & -1/3 & 2/9 \\ 1/3 & -1 & -1/3 \end{pmatrix}$$

$$M_{A \to B} = M_{\mathcal{E} \to B} \, M_{A \to \mathcal{E}}$$

$$= \begin{pmatrix} 4/9 & -1/3 & -1/9 \\ 1/9 & -1/3 & 2/9 \\ 1/3 & -1 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 5/9 & 1/9 \\ 0 & -1/9 & 7/9 \\ -1 & -1/3 & 1/3 \end{pmatrix}.$$

$$M_{B \to A} = M_{\mathcal{E} \to A} \, M_{B \to \mathcal{E}} =$$

$$= \begin{pmatrix} -1/2 & 1 & 1/2 \\ 3/4 & -1/2 & -1/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1/2 & -1 \\ 7/4 & -1/4 & 0 \\ 1/4 & 5/4 & 0 \end{pmatrix}.$$

Ex. 6.

$\phi_a : \mathbb{R}^3 \to \mathbb{R}^3$ t.c.

$\phi_a(x, y, z) = (x + ay, (1-a)y + z, ax + y + 2z)$

(a) $M_{\mathcal{E}\to\mathcal{E}}(\phi_a) = ?$

$$M_{\mathcal{E}\to\mathcal{E}}(\phi_a) = A_a = \begin{pmatrix} 1 & a & 0 \\ 0 & 1-a & 1 \\ a & 1 & 2 \end{pmatrix}$$

(b) $\det A_a = \begin{vmatrix} 1-a & 1 \\ 1 & 2 \end{vmatrix} - a \begin{vmatrix} 0 & 1 \\ a & 2 \end{vmatrix} =$

$= 2 - 2a - 1 + a^2 = a^2 - 2a + 1 = (a-1)^2$

NON SURIETT. $\Longleftrightarrow \boxed{a=1}$.

(c) $A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\Longrightarrow \operatorname{Im} A_1 = \langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{\text{BASE}} \rangle$

$\operatorname{Ker} A_1 = ?$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} I \\ II \\ \frac{III - I}{2} \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{cases} x+y=0 \to y=-x \\ z=0 \end{cases}$$

$\operatorname{Ker} A_1 = \langle \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\text{BASE}} \rangle$.

(d) $\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}.$

$$M_{\mathcal{A}\to\mathcal{A}}(\phi_a) = \,?$$

$$M_{\mathcal{A}\to\mathcal{A}}(\phi_a) = \underbrace{M_{\mathcal{E}\to\mathcal{A}}(\mathrm{id}_{\mathbb{R}^3})}\; \overbrace{M_{\mathcal{E}\to\mathcal{E}}(\phi_a)}^{(a)}\; \underbrace{M_{\mathcal{A}\to\mathcal{E}}(\mathrm{id}_{\mathbb{R}^3})}_{=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}}$$

$$\left(M_{\mathcal{A}\to\mathcal{E}}\right)^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} =$$

$$= \frac{1}{\det} S^T = \begin{pmatrix} -1/2 & 1 & 1/2 \\ 3/4 & -1/2 & -1/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}$$

$$\det = 2 - 2 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 - 2(2+1) \overset{!}{=} -4.$$

$$S = \begin{pmatrix} 2 & -3 & -1 \\ -4 & 2 & 2 \\ -2 & 1 & -1 \end{pmatrix} \qquad S^T = \begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1/2 & 1 & 1/2 \\ 3/4 & -1/2 & -1/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1-a & 1 \\ a & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} & & \\ & \cdot & \\ & & \end{pmatrix} \begin{pmatrix} 1+a & 2+a & -a \\ 2-a & 1-a & a+1 \\ a+3 & 2a+1 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} -a+3 & \frac{-a+1}{2} & \frac{3a+5}{2} \\ a-1 & \frac{3a+3}{4} & \frac{-5a-5}{4} \\ a & \frac{5a+1}{4} & \frac{-3a+1}{4} \end{pmatrix} = M_{\mathcal{A}\to\mathcal{A}}(\phi_a).$$

$$-\frac{1}{2}(1+a) + 2-a + \frac{1}{2}(a+3) =$$

$$= -\frac{1}{2} - \cancel{\frac{1}{2}a} + 2 - a + \cancel{\frac{1}{2}a} + \frac{3}{2} = -a+3$$

<u>Ex. 1</u>.

$A \in M_2(\mathbb{R})$

$\to$ AUTOVETTORI:

$w_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\lambda_1 = 0 \qquad \lambda_2 = 2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- $A w_1 = \lambda_1 w_1$ (1)
- $A w_2 = \lambda_2 w_2$ (2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad (2)$$

$$\begin{cases} 3a + b = 0 & \longrightarrow b = -3a \\ 3c + d = 0 & \longrightarrow d = -3c \\ a - 3b = 2 & \longrightarrow a + 9a = 2 \\ c - 3d = -6 & \longrightarrow c + 9c = -6 \end{cases}$$

$$\begin{cases} a = 1/5 \\ c = -3/5 \\ b = -3/5 \\ d = 9/5 \end{cases} \implies A = \begin{pmatrix} 1/5 & -3/5 \\ -3/5 & 9/5 \end{pmatrix}$$

$$\longrightarrow D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad T = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

Ex. 3.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{pmatrix} =$$

$$= (\lambda - 1)^2 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda + 2)(\lambda - 4)$$

$\Rightarrow$ AUTOVALORI:

$$\lambda_1 = -2$$
$$\lambda_2 = 4$$

$\Rightarrow$ AUTOSPAZI:

$$\mathrm{Ker}(A + 2I) = \mathrm{Ker}\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \left\langle \overset{v_1}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \right\rangle = A_{-2}$$

$$\mathrm{Ker}(A - 4I) = \mathrm{Ker}\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} = \left\langle \underset{v_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\rangle = A_4$$

$\Rightarrow$ DIAGONALIZZABILE.

$$D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

$$F = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$\longrightarrow$ AUTOVALORI:

$\lambda_1 = 0 \quad \mu_1 = 1$

$\lambda_2 = 1 \quad \mu_2 = 2 \quad g_2 \overset{?}{=} 2$

$$\mathrm{Ker}(F - I) = \mathrm{Ker}\begin{pmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \Big\langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{v_{1,1}}, \underbrace{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}}_{v_{1,2}} \Big\rangle = A_1$$

$-y - 2z = 0$

$\Rightarrow y = -2z \qquad\qquad \Rightarrow g_2 = 2.$

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$\Rightarrow$ DIAGONALIZZABILE

$$\mathrm{Ker} F = \mathrm{Ker}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \Big\langle \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{v_0} \Big\rangle$$

$$\begin{cases} z = 0 \\ x + 2y = 0 \longrightarrow x = -2y \end{cases}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D = T^{-1} A T.$$