TUTORATO 28/04.

FOGLIO 5.

Ex. 2

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$? \, k \to \exists \, \phi : \mathbb{R}^3 \to \mathbb{R}^4$ t.c.

- $\phi(v_1) = (1, 0, k, 0)$
- $\phi(v_2) = (1, 1, 0, 0)$
- $\phi(v_3) = (-1, k-3, 2-k, 1-k)$
- $\phi(v_4) = (2, k+1, 0, 0)$

$$a v_1 + b v_2 + c v_3 + d v_4 = 0$$

$$\begin{cases} a - b + 3c = 0 \\ 2a + b + d = 0 \\ a + b - c - d = 0 \end{cases}$$

$$\begin{cases} b = -2a \\ c = -a \\ d = 0 \end{cases}$$

$\to a = 1$

$v_1 - 2v_2 - v_3 = 0 \Rightarrow v_1 = 2v_2 + v_3$

Deve valere:

$\phi(v_1) = 2\phi(v_2) + \phi(v_3)$

$$\begin{pmatrix} 1 \\ 0 \\ k \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ k-3 \\ 2-k \\ 1-k \end{pmatrix}$$

$1 = 1$

$0 = 2 + k - 3 \quad 0 = 0$

$k = 2 - k \to 1 = 1 \quad \Rightarrow \boxed{k = 1}$

$0 = 1 - k \to k = 1$

(b) BASE DI $\mathbb{R}^3$ $V = \{v_2, v_3, v_4\}$

$A_{V \to \mathcal{E}}$

$A_{\mathcal{E} \to \mathcal{E}}$

$$\begin{cases} v_2 = -e_1 + e_2 + e_3 \\ v_3 = 3e_1 - e_3 \longrightarrow e_3 = 3e_1 - v_3 \\ v_4 = e_2 - e_3 \longrightarrow v_4 = e_2 - 3e_1 + v_3 \end{cases}$$

$$\hookrightarrow e_2 = 3e_1 - v_3 + v_4$$

$$v_2 = -e_1 + 3e_1 - v_3 + v_4 + 3e_1 - v_3$$

$$5e_1 = v_2 + 2v_3 - v_4$$

$$e_1 = \frac{1}{5}v_2 + \frac{2}{5}v_3 - \frac{1}{5}v_4$$

$$e_2 = \frac{3}{5}v_2 + \frac{6}{5}v_3 - \frac{3}{5}v_4 - v_3 + v_4 = \frac{3}{5}v_2 + \frac{1}{5}v_3 + \frac{2}{5}v_4$$

$$e_3 = \frac{3}{5}v_2 + \frac{6}{5}v_3 - \frac{3}{5}v_4 - v_3 = \frac{3}{5}v_2 + \frac{1}{5}v_3 - \frac{3}{5}v_4$$

$$\phi(e_1) = \frac{1}{5}\phi(v_2) + \frac{2}{5}\phi(v_3) - \frac{1}{5}\phi(v_4) = \frac{1}{5}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{2}{5}\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix} - \frac{1}{5}\begin{pmatrix}2\\2\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix} 1/5 - 2/5 - 2/5 \\ 1/5 - 4/5 - 2/5 \\ 2/5 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -1 \\ 2/5 \\ 0 \end{pmatrix}$$

$$\phi(e_2) = \frac{3}{5}\phi(v_2) + \frac{1}{5}\phi(v_3) + \frac{2}{5}\phi(v_4)$$

$$= \begin{pmatrix} 3/5 \\ 3/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/5 \\ -2/5 \\ 1/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6/5 \\ 1 \\ 1/5 \\ 0 \end{pmatrix}$$

$$\phi(e_3) = \frac{3}{5}\phi(v_2) + \frac{1}{5}\phi(v_3) - \frac{3}{5}\phi(v_4)$$

$$= \begin{pmatrix} 3/5 \\ 3/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/5 \\ -2/5 \\ 1/5 \\ 0 \end{pmatrix} - \begin{pmatrix} 6/5 \\ 6/5 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4/5 \\ -1 \\ 1/5 \\ 0 \end{pmatrix}$$

$$\Rightarrow A^{\xi}_{\xi}(\phi) = \begin{pmatrix} -3/5 & 6/5 & -4/5 \\ -1 & 1 & -1 \\ 2/5 & 1/5 & 1/5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(c)\quad \begin{cases} -\frac{3}{5}a + \frac{6}{5}b - \frac{4}{5}c = 0 \\ -a + b - c = 0 \\ \frac{2}{5}a + \frac{1}{5}b + \frac{1}{5}c = 0 \end{cases}$$

$$\begin{cases} -3a + 6b - 4c = 0 \\ c = -a + b \\ 2a + b + c = 0 \end{cases}$$

$$\begin{cases} -3a + 6b + 4a - 4b = 0 \\ c = -a + b = 3b \\ 2a + b - a + b = 0 \rightarrow a = -2b \end{cases}$$

$$6b + 6b - 8b - 4b = 0 \rightarrow 0 = 0$$

$$\begin{cases} b \text{ qls} \\ a = -2b \\ c = 3b \end{cases}$$

$$\rightarrow b = 1,\ a = -2,\ c = 3$$

$$-2\phi(e_1) + \phi(e_2) + 3\phi(e_3) = 0 \qquad \Rightarrow \boxed{rk = 2}.$$

$$\Rightarrow \phi(e_2) = 2\phi(e_1) - 3\phi(e_3).$$

FOGLIO 7.

Ex. 1.

$$\underset{2\times 2}{\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}} \underset{2\times 4}{\begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}} \underset{4\times 3}{\begin{pmatrix} 3 & -2 & 1 \\ 0 & 0 & 2 \\ 1 & -1 & 4 \\ 5 & 1 & -3 \end{pmatrix}} =$$

$$= \underset{2\times 3}{\begin{pmatrix} 17 & -9 & 37 \\ -5 & 4 & -17 \end{pmatrix}}.$$

Ex. 2.

$$\underbrace{\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a & b & c & d \\ x & y & z & w \end{pmatrix}}_{X^{2\times 4}} = \underbrace{\begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}}_{B}$$

$$AX = B \longrightarrow \overbrace{A^{-1}A}^{I_{2\times 2}}X = A^{-1}B$$

$$\Rightarrow X = A^{-1}B.$$

$$A^{-1} = \frac{1}{\underbrace{\det A}_{=1}} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 & -1 \\ 1 & 2 & 6 & 2 \end{pmatrix}.$$

Ex. 5.

$k \in \mathbb{R}$

$$A_k = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & k \\ 1+k & -k & 1 \end{pmatrix}$$

$$\det(A_k) = 1 \cdot \begin{vmatrix} -1 & k \\ -k & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & -1 \\ 1+k & -k \end{vmatrix} =$$

$$= \cancel{-1} + k^2 - 2k \cancel{+1} + k =$$

$$= k^2 - k = k(k-1).$$

$$\det(A_k) = 0 \iff \underbrace{k = 0 \lor k = 1}_{A_k \text{ NON INVERTIBILE}}$$

$$\boxed{k \neq 0,1} \Rightarrow A_k \text{ INVERTIBILE.}$$

$$S = \begin{pmatrix} k^2-1 & k^2+k-2 & -k+1 \\ -k & -k & k \\ 1 & -k+2 & -1 \end{pmatrix}$$

$$S_{11} = (-1)^{1+1} \begin{vmatrix} -1 & k \\ -k & 1 \end{vmatrix} = -1 + k^2$$

$$S_{12} = (-1)^{1+2} \begin{vmatrix} 2 & k \\ 1+k & 1 \end{vmatrix} = -(2 - k(1+k)) = -2 + k + k^2$$

$$S_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 1+k & -k \end{vmatrix} = -2k + 1 + k = -k + 1$$

$$\delta_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 1 \\ -k & 1 \end{vmatrix} = -k$$

$$\delta_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1+k & 1 \end{vmatrix} = 1 - (1+k) \overset{!}{=} -k$$

$$A_k^{-1} = \frac{1}{\det(A_k)} \delta^T$$

$$= \frac{1}{k(k-1)} \begin{pmatrix} k^2-1 & -k & 1 \\ \overbrace{k^2+k-2}^{(k+2)(k-1)} & -k & -k+2 \\ -k+1 & k & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k+1}{k} & \frac{-1}{k-1} & \frac{1}{k(k-1)} \\ \frac{k+2}{k} & -\frac{1}{k-1} & \frac{-k+2}{k(k-1)} \\ \frac{-k+1}{k(k-1)} & \frac{1}{k-1} & -\frac{1}{k(k-1)} \end{pmatrix}$$

$$\frac{-(\cancel{k-1})}{k(\cancel{k-1})} = -\frac{1}{k}$$

Ex. 6.

$k \in \mathbb{R}$

$$\begin{cases} x + z = 2 \\ 2x - y + kz = 1 \\ (1+k)x - ky + z = 2 \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & k \\ 1+k & -k & 1 \end{pmatrix}}_{A_k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\det A_k = \begin{vmatrix} -1 & k \\ -k & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1+k & -k \end{vmatrix} = k(k-1)$$

$$\det A_k \neq 0 \iff \boxed{k \neq 0, 1}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_k^{-1} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{k+1}{k} & \frac{-1}{k-1} & \frac{1}{k(k-1)} \\ \frac{k+2}{k} & -\frac{1}{k-1} & \frac{-k+2}{k(k-1)} \\ -\frac{1}{k} & \frac{1}{k-1} & -\frac{1}{k(k-1)} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{2k+2}{k} - \frac{1}{k-1} + \frac{2}{k(k-1)} \\ \frac{2k+4}{k} - \frac{1}{k-1} + \frac{-2k+4}{k(k-1)} \\ -\frac{2}{k} + \frac{1}{k-1} - \frac{2}{k(k-1)} \end{pmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$1 = \frac{2\overbrace{(k+1)(k-1)}^{k^2-1} - k + 2}{k(k-1)}$$

$$= \frac{2k^2 - k}{k(k-1)} = \frac{\cancel{k}(2k-1)}{\cancel{k}(k-1)} = \frac{2k-1}{k-1}$$

$$2 = \frac{2(k+2)(k-1) - k - 2k + 4}{k(k-1)} =$$

$$= \frac{2(k^2+k-2)-3k+4}{k(k-1)} = \frac{2k^2-k}{k(k-1)} =$$

$$= \frac{\cancel{k}(2k-1)}{\cancel{k}(k-1)} = \frac{2k-1}{k-1}$$

3) $$= \frac{-2(k-1)+k-2}{k(k-1)} = -\frac{1}{k-1}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2k-1}{k-1} \\ \frac{2k-1}{k-1} \\ -\frac{1}{k-1} \end{pmatrix} \quad k \neq 0, 1$$

Ex. 7.

$k \in \mathbb{R}$

$$\begin{cases} -x+ky+2z=1 \\ -kx+y+(1+k)z=2 \\ y+z=2 \end{cases}$$

$$\underbrace{\begin{pmatrix} -1 & k & 2 \\ -k & 1 & 1+k \\ 0 & 1 & 1 \end{pmatrix}}_{A_k} \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}_{b}$$

$\to \det A_k = 0$

$\to rk(A_k) < rk(A_k|b)$

$$\det A_k = - \begin{vmatrix} -1 & 2 \\ -k & 1+k \end{vmatrix} + \begin{vmatrix} -1 & k \\ -k & 1 \end{vmatrix} =$$

$$= -(-1-k+2k) + (-1+k^2) =$$

$$= k^2 - k = k(k-1)$$

$$\det A_k = 0 \iff k=0 \lor k=1.$$

• $\boxed{k=0}$.

$$A_0 = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$III = -2I + II$

$$\mathrm{Im}(A_0) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \overset{?}{\in} \mathrm{Im}(A_0)$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \overset{?}{=} a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} a = 1 \\ b = 2 \\ b = 2 \end{cases} \implies \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \in \mathrm{Im}(A_0)$$

$\implies$ INF. SOL.

$\boxed{k = 1}$

$$A_1 = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathrm{Im}(A_1) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \overset{?}{\in} \mathrm{Im}(A_1)$$

$$\begin{cases} a + b = 1 \\ a + b = 2 \\ b = 2 \end{cases} \qquad \begin{array}{l} a = -1 \\ -1 + 2 \neq 2 \\ b = 2 \end{array}$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \notin \mathrm{Im}(A_1) \Longrightarrow \text{NON HA SOL.}$$

$\boxed{k = 1}$.

Ex. 8.

$k \in \mathbb{R}$

$$\begin{cases} 2x + ky - z = 1 \\ x + y = 2 \\ (1+k)x + y - kz = 2 \end{cases}$$

$$\underbrace{\begin{pmatrix} 2 & k & -1 \\ 1 & 1 & 0 \\ 1+k & 1 & -k \end{pmatrix}}_{A_k} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}_{b}$$

$$\det A_k = - \begin{vmatrix} k & -1 \\ 1 & -k \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1+k & -k \end{vmatrix} =$$

$$= k^2 - \cancel{1} - 2k + \cancel{1} + k =$$

$$= k^2 - k = k(k-1)$$

$$\Rightarrow \det A_k = 0 \iff k = 0 \lor k = 1.$$

• $\boxed{k=0}$.

$$A_0 = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$I_c = -2III_c + I_c$

$$\operatorname{Im}(A_0) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \overset{?}{\in} \operatorname{Im}(A_0)$$

$$\begin{aligned} 1 &= a \\ 2 &= b \\ 2 &= b \end{aligned} \implies \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \in \operatorname{Im}(A_0)$$

$\implies$ INF. SOL.

$$\left(\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \end{array}\right) \quad \begin{array}{l} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{l} II \\ I \\ III - II \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} I \\ II - 2I \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x+y=2 \rightarrow x = 2-y \\ y+\frac{1}{2}z = \frac{3}{2} \rightarrow z = 3-2y \end{cases}$$

$\hookrightarrow 2y+z=3$

$$S = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\rangle, \ k=0.$$

- $\boxed{k=1}$.

$$A_1 = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\mathrm{Im}(A_1) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix}1\\2\\2\end{pmatrix} \overset{?}{\in} \operatorname{Im}(A_1)$$

$$\begin{aligned}1 &= a+b \longrightarrow b=-1\\ 2 &= a \longrightarrow a=2\\ 2 &= a+b \quad 2-1 \neq 2\end{aligned}$$

$\Longrightarrow$ NO SOL ! $\quad S = \emptyset$.

INF. SOL. $\Longleftrightarrow \boxed{k=1}$.

Ex. 4.

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\1\end{pmatrix} \qquad A = \begin{pmatrix}1 & 0 & 1\\ 2 & -1 & 0\\ -1 & -1 & -2\end{pmatrix}$$

$$\det A = \begin{vmatrix}-1 & 0\\ -1 & -2\end{vmatrix} + \begin{vmatrix}2 & -1\\ -1 & -1\end{vmatrix} =$$

$$= \cancel{2} - \cancel{2} - 1 = -1 \neq 0.$$

$\Rightarrow$ 1 sola sol.

$$A^{-1} = \frac{1}{\det A} S^{T}$$

$$S = \begin{pmatrix} 2 & 4 & -3 \\ -1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2 & 1 & -1 \\ -4 & 1 & -2 \\ 3 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 \\ -4 & 1 & -2 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ -11 \\ 8 \end{pmatrix}.$$