TUTORATO 13/04.

Ex. 1

$\phi : \mathbb{R}^4 \to \mathbb{R}^3$

$$\phi(x,y,z,w) = \begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix}$$

(a) $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$

- $\phi(v_1+v_2) = \phi(v_1)+\phi(v_2)$
- $\phi(\lambda v) = \lambda\phi(v)$ $\quad v=(x,y,z,w)$

$$\phi(v_1+v_2) = \phi\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ w_1+w_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2+y_1+y_2+2(z_1+z_2) \\ y_1+y_2-(z_1+z_2)+w_1+w_2 \\ 2(x_1+x_2)+y_1+y_2-(w_1+w_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1+y_1+2z_1 \\ y_1-z_1+w_1 \\ 2x_1+y_1-w_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2+2z_2 \\ y_2-z_2+w_2 \\ 2x_2+y_2-w_2 \end{pmatrix} = \phi(v_1)+\phi(v_2)$$

$$\phi(\lambda v) = \phi\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{pmatrix} = \begin{pmatrix} \lambda x+\lambda y+2\lambda z \\ \lambda y-\lambda z+\lambda w \\ 2\lambda x+\lambda y-\lambda w \end{pmatrix} = \lambda\phi(v)$$

$\Longrightarrow \phi$ LINEARE

(b) $v \in \operatorname{Ker} \phi \Longleftrightarrow \phi(v) = 0$

$$\begin{cases} x + y + 2z = 0 \\ y - z + w = 0 \longrightarrow z = y + w = 2x + 2y \\ 2x + y - w = 0 \longrightarrow w = 2x + y \end{cases}$$

$$x + y + 4x + 4y = 0$$

$$5x = -5y \longrightarrow y = -x$$

$$\begin{cases} y = -x \\ z = 0 \\ w = x \end{cases} \qquad \operatorname{Ker} \phi = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\dim(\mathbb{R}^4) = 4 = \underbrace{\dim \operatorname{Ker} \phi}_{\substack{\| \\ 1}} + \dim \operatorname{Im} \phi \Longrightarrow \dim \operatorname{Im} \phi = 3$$

$$\phi \longleftrightarrow A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix} = M(\phi)^{\mathcal{E}}$$

$$\operatorname{Im} \phi = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

(c)

$$\mathcal{V} = \left\{ \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, & \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, & \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \end{array} \right\}$$ DOMINIO

$$\mathcal{W} = \left\{ \begin{array}{ccc} \begin{pmatrix}1\\2\\1\end{pmatrix}, & \begin{pmatrix}-1\\1\\1\end{pmatrix}, & \begin{pmatrix}0\\0\\1\end{pmatrix} \\ w_1 & w_2 & w_3 \end{array} \right\}$$ CODOMINIO

$M(\phi)^{w}_{v} = ?$

$\phi(e_1) = e_1 + 2e_3 \leftarrow$

$\phi(e_2) = e_1 + e_2 + e_3 \leftarrow$

$\phi(e_3) = 2e_1 - e_2$

$\phi(e_4) = e_2 - e_3$

$v_1 = e_1 + e_2$

$v_2 = e_2 + e_3$

$v_3 = e_3 + e_4$

$v_4 = e_2$

$\phi(v_1) = \phi(e_1 + e_2) = \phi(e_1) + \phi(e_2) =$

$= e_1 + 2e_3 + e_1 + e_2 + e_3 = 2e_1 + e_2 + 3e_3$

$\phi(v_2) = \phi(e_2 + e_3) = \phi(e_2) + \phi(e_3) =$

$= e_1 + \cancel{e_2} + e_3 + 2e_1 - \cancel{e_2} = 3e_1 + e_3$

$$\phi(v_3) = \phi(e_3 + e_4) = \phi(e_3) + \phi(e_4) =$$
$$= 2e_1 - \cancel{e_2} + \cancel{e_2} - e_3 = 2e_1 - e_3$$
$$\phi(v_4) = \phi(e_2) = e_1 + e_2 + e_3$$

$$\begin{cases} w_1 = e_1 + 2e_2 + e_3 \\ w_2 = -e_1 + e_2 + e_3 \\ w_3 = e_3 \end{cases}$$

$$\begin{cases} e_3 = w_3 \\ e_1 = e_2 + w_3 - w_2 \\ w_1 = e_2 + w_3 - w_2 + 2e_2 + w_3 \end{cases}$$

$$\rightarrow 3e_2 = w_1 + w_2 - 2w_3$$
$$\rightarrow e_2 = \frac{1}{3}w_1 + \frac{1}{3}w_2 - \frac{2}{3}w_3$$
$$\rightarrow e_1 = \frac{1}{3}w_1 - \frac{2}{3}w_2 + \frac{1}{3}w_3$$

$$\phi(v_1) = 2e_1 + e_2 + 3e_3$$
$$\overset{!}{=} \frac{2}{3}w_1 - \frac{4}{3}w_2 + \cancel{\frac{2}{3}w_3} + \frac{1}{3}w_1 + \frac{1}{3}w_2 - \cancel{\frac{2}{3}w_3} + 3w_3$$
$$= w_1 - w_2 + 3w_3$$

$$\phi(v_2) = 3e_1 + e_3$$
$$\overset{!}{=} w_1 - 2w_2 + w_3 + w_3 = w_1 - 2w_2 + 2w_3$$

$$\phi(v_3) = 2e_1 - e_3$$
$$= \frac{2}{3}w_1 - \frac{4}{3}w_2 + \frac{2}{3}w_3 - w_3$$
$$= \frac{2}{3}w_1 - \frac{4}{3}w_2 - \frac{1}{3}w_3$$

$$\phi(v_4) = e_1 + e_2 + e_3$$
$$= \frac{1}{3}w_1 - \frac{2}{3}w_2 + \frac{1}{3}w_3 + \frac{1}{3}w_1 + \frac{1}{3}w_2 - \frac{2}{3}w_3 + $$
$$+ w_3$$
$$= \frac{2}{3}w_1 - \frac{1}{3}w_2 + \frac{2}{3}w_3$$

$$M(\phi)^W_V = \begin{pmatrix} 1 & 1 & 2/3 & 2/3 \\ -1 & -2 & -4/3 & -1/3 \\ 3 & 2 & -1/3 & 2/3 \end{pmatrix}$$

(d) ? $\exists\, W \subset \mathbb{R}^4$ t.c. $\dim(\phi(W)) = \dim(W)$

$W = \langle e_1, e_2, e_3 \rangle$ $\dim W = 3$

$$\phi(e_1) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \phi(e_2) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \phi(e_3) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\phi(W) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

$\dim(\phi(W)) = 3$.

Ex. 3

$$U: \begin{cases} x+y+z+w=0 \\ 2x+3y-z+w=0 \end{cases}$$

$$V = \left\langle \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{v_3} \right\rangle$$

(a) ¿k

$\exists\, \phi : \mathbb{R}^4 \to \mathbb{R}^2$ t.c.

- $\ker \phi = U \leftarrow$
- $\phi(v_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- $\phi(v_2) = \begin{pmatrix} -2 \\ -k \end{pmatrix}$
- $\phi(v_3) = \begin{pmatrix} 1 \\ k \end{pmatrix}$

$$U: \begin{cases} z = 2x+3y+w \\ x+y+2x+3y+w+w=0 \end{cases}$$

$$3x+4y+2w=0$$

$$\begin{cases} w = -\frac{3}{2}x - 2y \\ z = \frac{1}{2}x + y \end{cases}$$

$$U = \left\langle \underset{u_1}{\begin{pmatrix} 2 \\ 0 \\ 1 \\ -3 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix}} \right\rangle$$

$\phi(u_1) = \underline{0}$

$\phi(u_2) = \underline{0}$

$a u_1 + b u_2 + c v_1 + d v_2 + e v_3 = 0$

$$\begin{cases} 2a + c + d = 0 \\ b - c = 0 \longrightarrow c = b \\ a + b + c - d + e = 0 \\ -3a - 2b - c + e = 0 \end{cases}$$

$$\begin{cases} d = -2a - b \\ c = b \\ a + b + b + 2a + b + e = 0 \\ -3a - 2b - b + e = 0 \rightarrow e = 3a + 3b \end{cases}$$

$3a + 3b + 3a + 3b = 0 \rightarrow b = -a$

$$\begin{cases} a \text{ qls} \\ b = -a \\ c = -a \\ d = -a \\ e = 0 \end{cases}$$

$u_1 - u_2 - v_1 - v_2 = 0$

$$\boxed{u_1 = u_2 + v_1 + v_2}$$

$$\phi(u_1) = \phi(u_2 + v_1 + v_2) = \phi(u_2) + \phi(v_1) + \phi(v_2)$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -k \end{pmatrix}$$

$$\begin{cases} 2 = 2 \\ 1 - k = 0 \end{cases} \Longleftrightarrow \boxed{k = 1}$$

(b) $A = M(\phi)_{\mathcal{E}}^{\mathcal{E}} = ?$

$\phi(e_1) = ?$ $\phi(e_2) = ?$ $\phi(e_3) = ?$ $\phi(e_4) = ?$

$$\begin{cases} u_2 = e_2 + e_3 - 2e_4 \\ v_1 = e_1 - e_2 + e_3 - e_4 \\ v_2 = e_1 - e_3 \\ v_3 = e_3 + e_4 \end{cases}$$

$\rightarrow e_1 = \frac{1}{6} u_2 + \frac{1}{6} v_1 + \frac{5}{6} v_2 + \frac{1}{2} v_3$

$\rightarrow e_2 = \frac{1}{2} u_2 - \frac{1}{2} v_1 + \frac{1}{2} v_2 + \frac{1}{2} v_3$

$\rightarrow e_3 = \frac{1}{6} u_2 + \frac{1}{6} v_1 - \frac{1}{6} v_2 + \frac{1}{2} v_3$

$\rightarrow e_4 = -\frac{1}{6} u_2 - \frac{1}{6} v_1 + \frac{1}{6} v_2 + \frac{1}{2} v_3$

$$\phi(e_1) = \frac{1}{6}\phi(u_2) + \frac{1}{6}\phi(v_1) + \frac{5}{6}\phi(v_2) + \frac{1}{2}\phi(v_3)$$

$$= \frac{1}{6}\begin{pmatrix}0\\0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}2\\1\end{pmatrix} + \frac{5}{6}\begin{pmatrix}-2\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1/3\\1/6\end{pmatrix} + \begin{pmatrix}-5/3\\-5/6\end{pmatrix} + \begin{pmatrix}1/2\\1/2\end{pmatrix} = \begin{pmatrix}-5/6\\-1/6\end{pmatrix}$$

$$\phi(e_2) = \frac{1}{2}\phi(u_2) - \frac{1}{2}\phi(v_1) + \frac{1}{2}\phi(v_2) + \frac{1}{2}\phi(v_3)$$

$$= \begin{pmatrix}-3/2\\-1/2\end{pmatrix}$$

$$\phi(e_3) = \frac{1}{6}\phi(u_2) + \frac{1}{6}\phi(v_1) - \frac{1}{6}\phi(v_2) + \frac{1}{2}\phi(v_3)$$

$$= \begin{pmatrix}7/6\\5/6\end{pmatrix}$$

$$\phi(e_4) = -\frac{1}{6}\phi(u_2) - \frac{1}{6}\phi(v_1) + \frac{1}{6}\phi(v_2) + \frac{1}{2}\phi(v_3)$$

$$= \begin{pmatrix}-1/6\\1/6\end{pmatrix}$$

$$A = \begin{pmatrix}-5/6 & -3/2 & 7/6 & -1/6\\ -1/6 & -1/2 & 5/6 & 1/6\end{pmatrix}$$

(c) $\operatorname{rk} A = ?$

$$\dim \mathbb{R}^4 = 4 = \underbrace{\dim \operatorname{Ker}\phi}_{\displaystyle \| \atop \displaystyle 2} + \underbrace{\dim \operatorname{Im}\phi}_{\displaystyle \| \atop \displaystyle \operatorname{rk}A}$$

$$\Longrightarrow \operatorname{rk} A = 2 .$$

(d) $\phi^{-1}(\{(1,2)\}) = ?$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = A v \quad , \quad v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{cases} -\frac{5}{6}x_1 - \frac{3}{2}x_2 + \frac{7}{6}x_3 - \frac{1}{6}x_4 = 1 \\ -\frac{1}{6}x_1 - \frac{1}{2}x_2 + \frac{5}{6}x_3 + \frac{1}{6}x_4 = 2 \end{cases}$$

$$\begin{cases} -5x_1 - 9x_2 + 7x_3 - x_4 = 6 \\ -x_1 - 3x_2 + 5x_2 + x_4 = 12 \end{cases}$$

$\to x_1 = 15 - 4x_3 - 2x_4$

$\to x_2 = -9 + 3x_3 + x_4$

$x_3 = \text{qls}$

$x_4 = \text{qls}$

$$\Rightarrow \phi^{-1}\left(\left\langle \begin{pmatrix}1\\2\end{pmatrix}\right\rangle\right) = \begin{pmatrix}15\\-9\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}-4\\3\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\0\\1\end{pmatrix}\right\rangle$$

<u>Ex. 5</u>

$$A_k = \begin{array}{c} \begin{pmatrix} 1 & 2-k & 1 \\ -1 & -k & -3 \\ k & 1 & k+1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ a & b & c \end{matrix} \end{array}$$

(a)

$$\begin{cases} a+(2-k)b+c=0 \\ -a-kb-3c=0 \\ ka+b+(k+1)c=0 \end{cases}$$

$$\begin{cases} c=-a+(k-2)b \\ -a-kb+3a-3kb+6b=0 \\ \cancel{ka}+b+(k+1)(k-2)b\underbrace{-(k+1)a}_{\cancel{-ka}-a}=0 \end{cases}$$

$\rightarrow 2a=4kb-6b$

$\rightarrow a=2kb-3b=(2k-3)b$

$c=(3-2k+k-2)b=(1-k)b$

$$b+(k+1)(k-2)b-(2k-3)b=0$$

$$(k^2 - k - 2 - 2k + 3 + 1)b = 0$$

$$\underbrace{(k^2 - 3k + 2)}_{\neq 0} b = 0$$

$$k^2 - 3k + 2 = (k-1)(k-2) = 0 \iff k = 1 \lor k = 2$$

$$\operatorname{rk} A_k = 3 \quad \boxed{k \neq 1, 2}.$$

- $\boxed{k = 1}$

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -3 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\operatorname{Ker} A_1 : \begin{cases} x + y + z = 0 \\ -x - y - 3z = 0 \\ x + y + 2z = 0 \end{cases}$$

$$\begin{cases} x \text{ qls} \\ y = -x \\ z = 0 \end{cases} \implies \operatorname{Ker} A_1 = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

$$\operatorname{Im} A_1 = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right\rangle$$

$\boxed{k=2}$

$$A_2 = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -2 & -3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\ker A_2 : \begin{cases} x + z = 0 \\ -x - 2y - 3z = 0 \\ 2x + y + 3z = 0 \end{cases} \qquad \begin{cases} x \text{ qls} \\ y = x \\ z = -x \end{cases}$$

$$\ker A_2 = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$\operatorname{Im} A_2 = \left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

(c) $\boxed{k=0}$

$$A_0^{-1}(\{(1,0,0)\}) = ?$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A_0 v, \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x + 2y + z = 1 \\ -x - 3z = 0 \\ y + z = 0 \end{cases} \longrightarrow \begin{cases} x = 3/4 \\ y = 1/4 \\ z = -1/4 \end{cases}$$

$$\Rightarrow A_0^{-1}(\{(1,0,0)\}) = \begin{pmatrix} 3/4 \\ 1/4 \\ -1/4 \end{pmatrix}.$$

(d) $A_k^{-1}(\{(2,-4,5)\})$

$\boxed{k=1}$ $$\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} = A_k \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} \overset{?}{\in} \operatorname{Im} A_1$$

$$\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} \overset{?}{=} a \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$\begin{cases} a+b=2 & \to b = 2-a = 1 \\ -a-3b=-4 & \to -a-6+3a=-4 \\ a+2b=5 & \quad \hookrightarrow a=1 \end{cases}$$

$1+2=5$

$3 \neq 5$ NO $\Rightarrow$ IMP.

$$A_1^{-1}(\{(2,-4,5)\}) = \emptyset.$$

$\boxed{k=2}$

$$\begin{cases} a=2 \\ -a-2b=-4 \longrightarrow -2b=-2 \longrightarrow b=1 \\ 2a+b=5 \end{cases}$$

$4+1=5$ ✓

$\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix} \in \operatorname{Im} A_2$

$A_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ -1 & -2 & -3 & -4 \\ 2 & 1 & 3 & 5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -2 & -2 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} II+I \\ III-2I \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) II\left(-\frac{1}{2}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x+z=2$
$y+z=1$

$$\begin{cases} x=2-z \\ y=1-z \end{cases}$$

$$\Rightarrow A_2^{-1}\left(\{(2,-4,5)\}\right) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle .$$