TUTORATO 30/03/22

Ex. 2

$$U = \left\{ (x, y, t, u, v, w) \in \mathbb{R}^6 \;\middle|\; \begin{cases} x + y - t - u + v + w = 0 \\ 2x + y - t - 2u + 2w = 0 \\ x + 2t - u - v + w = 0 \\ 3x + y + 3t - 3u + v + w = 0 \end{cases} \right\}$$

$$\left(\begin{array}{cccccc|c} 1 & 1 & -1 & -1 & 1 & 1 & 0 \\ 2 & 1 & -1 & -2 & 0 & 2 & 0 \\ 1 & 0 & 2 & -1 & -1 & 1 & 0 \\ 3 & 1 & 3 & -3 & 1 & 1 & 0 \end{array}\right)$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -2 & 0 \\ 0 & -1 & 3 & 0 & -2 & 0 \\ 0 & -2 & 6 & 0 & -2 & -2 \end{pmatrix} & \begin{array}{l} \mathrm{I} \\ \mathrm{II} - 2\mathrm{I} \\ \mathrm{III} - \mathrm{I} \\ \mathrm{IV} - 3\mathrm{I} \end{array} \end{array}$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 & -2 & 0 \\ 0 & -2 & 6 & 0 & -2 & -2 \end{pmatrix} & \begin{array}{l} \mathrm{I} \\ -\mathrm{II} \\ \\ \\ \end{array} \end{array}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & -2 \end{pmatrix} \begin{array}{l} \\ \\ III + II \\ IV + 2II \end{array}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{array}{l} \\ \\ III/2 \\ \dfrac{IV - 2III}{2} \end{array}$$

$$\begin{cases} v = w \\ z = 0 \\ y - \cancel{z} + 2v = 0 \\ x + y - \cancel{z} - u + v + w = 0 \end{cases}$$

$$\begin{cases} v = w \\ z = 0 \\ y = -2w \\ u = x + \cancel{y} + \cancel{2w} = x \end{cases}$$

$$\text{BASE DI } U = \left\{ \underset{u_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}} \right\} \Rightarrow \dim U = 2.$$

$$\text{BASE DI } \mathbb{R}^6 = \{u_1, u_2, e_1, e_2, e_3, e_4\}.$$

Ex. 3.

$\mathbb{R}^4$

$$V = \left\langle \underset{v_1}{\begin{pmatrix}1\\2\\1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\1\\2\\1\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}-1\\1\\-4\\-3\end{pmatrix}} \right\rangle \quad v_3 = 2v_1 - 3v_2$$

$$\begin{cases} x = a + b - c \\ y = 2a + b + c \\ z = a + 2b - 4c \\ \mu = b - 3c \end{cases}$$

$$\begin{cases} c = a + b - x \\ y = 2a + b + a + b - x = 3a + 2b - x \\ z = a + 2b - 4a - 4b + 4x \\ \mu = b - 3a - 3b + 3x \end{cases}$$

$$\begin{cases} \overline{x + y = 3a + 2b} \\ z = -3a - 2b + 4x \\ \mu = -2b - 3a + 3x \end{cases}$$

$$\begin{cases} \overline{b = \frac{1}{2}x + \frac{1}{2}y - \frac{3}{2}a} \\ z = -\cancel{3a} - x - y + \cancel{3a} + 4x = 3x - y \end{cases}$$

$$| \; u = -x - y + \cancel{3a} - \cancel{3a} + 3x = 2x - y$$

$$\begin{cases} 3x - y - t = 0 \longrightarrow t = 3x - y \\ 2x - y - u = 0 \longrightarrow u = 2x - y \end{cases}$$

$\Rightarrow \dim V = 2$.

$V' = ?$

$\dim V' = \dim V + 1 = 3 \quad V \subseteq V'$

BASE DI $V = \{ v_1, v_2 \}$

$V' = \langle v_1, v_2, e_1 \rangle$

$$\begin{cases} x = a + b + c \\ y = 2a + b \\ z = a + 2b \\ u = b \end{cases}$$

$$\begin{cases} x = a + c + u \longrightarrow x = z - 2u + c + u \\ y = 2a + u \longrightarrow y = 2z - 4u + u = 2z - 3u \\ z = a + 2u \longrightarrow a = z - 2u \\ b = u \end{cases}$$

$$\begin{cases} c = x - z + u \\ y - 2z + 3u = 0 \\ a = z - 2u \\ b = u \end{cases} \Longrightarrow \boxed{y - 2z + 3u = 0}$$

Ex. 4.

$\mathcal{M}_{2,2}(\mathbb{R})$

$$W = \left\langle \underset{W_1}{\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}}, \underset{W_2}{\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}}, \underset{W_3}{\begin{pmatrix} -3 & 3 \\ -3 & 2 \end{pmatrix}}, \underset{W_4}{\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}}, \underset{W_5}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \right\rangle$$

$$\begin{cases} 2a + b - 3c + d = 0 \\ -a - b + 3c + d + e = 0 \\ a - b - 3c - d = 0 \\ 2c + 2d + e = 0 \end{cases}$$

$$\begin{cases} d = -2a - b + 3c \\ -a - b + 3c - 2a - b + 3c + e = 0 \\ a \cancel{-b} - 3c + 2a \cancel{+b} - 3c = 0 \end{cases}$$

$$\left| \; 2c - 4a - 2b + 6c + e = 0 \right.$$

$$\begin{cases} -3a - 2b + 6c + e = 0 \\ 3a - 6c = 0 \longrightarrow a = 2c \\ 3a - 6c = 0 \end{cases}$$

$$\begin{cases} d = -b - c \\ e = 2b \\ a = 2c \end{cases}$$

- $\boxed{b = 1, c = 0}$

$a = 0$
$b = 1$
$c = 0$
$d = -1$
$e = 2$

- $\boxed{b = 0, c = 1}$

$a = 2$
$b = 0$
$c = 1$
$d = -1$
$e = 0$

$$W_2 - W_4 + 2W_5 = 0 \longrightarrow W_4 = W_2 + 2W_5$$

$$2W_1 + W_3 - W_4 = 0 \longrightarrow W_3 = W_4 - 2W_1 = W_2 + 2W_5 - 2W_1$$

$$\Longrightarrow \dim W = 3$$

BASE DI $W = \{ W_1, W_2, W_5 \}$

$\dim(\mathcal{M}_{2,2}(\mathbb{R})) = 4$

$W \oplus W' = \mathcal{M}_{2,2}(\mathbb{R})$

$\uparrow$ dim 3

$\Longrightarrow \dim W' = 1.$

$$W' = \left\langle \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\overline{W}} \right\rangle$$

$$a W_1 + b W_2 + c W_3 + d \overline{W} = 0$$

$$\begin{cases} 2a + b + d = 0 \longrightarrow d = 0 \\ -a - b + c = 0 \longrightarrow -2a = 0 \longrightarrow a = 0 \\ a - b = 0 \longrightarrow a = b \longrightarrow b = 0 \\ c = 0 \end{cases}$$

$\Longrightarrow$ LIN. IND.

<u>EX.</u>

$T \subset \mathbb{R}^4$

$T = \langle v_1, v_2 \rangle$, $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ v_1, v_2 LIN. IND.

$\Longrightarrow \dim T = 2.$

$$S: \begin{cases} -x + 2y + 2z = 0 \\ -x + 2z + 2w = 0 \end{cases}$$

(d) $S \cap T = ?$

BASE DI $S+T = ?$

$w \in T$

$$w = a_1 v_1 + a_2 v_2 = \begin{pmatrix} a_1 + 3a_2 \\ a_1 \\ a_1 \\ 2a_1 - a_2 \end{pmatrix}$$

$$\begin{cases} -(a_1+3a_2)+2a_1+2a_1=0 \\ -(a_1+3a_2)+2a_1+2(2a_1-a_2)=0 \end{cases}$$

$$\begin{cases} -a_1-3a_2+4a_1=0 \\ -a_1-3a_2+2a_1+4a_1-2a_2=0 \end{cases}$$

$$\begin{cases} 3a_1=3a_2 \to a_1=a_2 \Rightarrow \boxed{a_1=a_2} \\ 5a_1=5a_2 \to a_1=a_2 \end{cases}$$

$$w = \begin{pmatrix} 4 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \dim(S \cap T) = 1$$

$S \cap T = \langle w \rangle$.

$$\dim(S+T) = \underset{2}{\dim(S)} + \underset{2}{\dim(T)} - \underset{1}{\dim(S \cap T)} = 2+2-1 = 3$$

$\dim T = 2$

$$S : \begin{cases} x = 2y + 2z \\ x = 2z + 2w \end{cases}$$

$$\cancel{2z} + 2w = 2y + \cancel{2z}$$

$$\begin{cases} y = w \\ x = 2z + 2w \end{cases} \implies \dim S = 2$$

$$S = \left\langle \underset{s_1}{\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{s_2}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}} \right\rangle$$

$S + T = \langle v_1, v_2, s_1 \rangle$. $\longleftarrow$

(b) $L \subseteq \mathbb{R}^4$ t.c. $\underbrace{(T+S)}_{\dim = 3} \oplus L = \underbrace{\mathbb{R}^4}_{\dim = 4}$

$$\implies \dim L = 1.$$

$L = \langle l \rangle \quad l = e_1$.

(c) $L_1 \subseteq \mathbb{R}^4$, $L_1 \neq L$

$(T+S) \oplus L_1 = \mathbb{R}^4$, $L \oplus L_1$

Può essere $L \oplus L_1 = \mathbb{R}^4$? NO!

$\dim L_1 = 1 \quad L_1 = \langle l_1 \rangle$

$\begin{cases} v_1, v_2, s_1, e_1 \text{ LIN. IND.} \\ e, e_1 \text{ LIN. IND.} \end{cases}$

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_4 .$$

$$\underbrace{L}_{\dim = 1} \oplus \underbrace{L_1}_{\dim 1} \neq \underbrace{\mathbb{R}^4}_{\dim 4}$$

(d) $M \subseteq \mathbb{R}^4$, $M \neq S$, $\dim M = 2$

$T + S = T + M$

$T + S = \langle v_1, v_2, s_1 \rangle$

$M = \langle s_1, v_2 \rangle \Rightarrow \dim M = 2$

$M \neq S = \langle s_1, s_2 \rangle$

in quanto M non contiene s_2.

$T + M = \langle v_1, v_2, s_1 \rangle = T + S$.

$T = \langle v_1, v_2 \rangle \quad M = \langle v_2, s_1 \rangle$

$\Rightarrow T + M = \langle \underbrace{v_1, v_2, s_1}, \rangle = T + S$

$$s_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \stackrel{?}{=} a v_2 + b s_1$$

$$= a \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$2 = 3a + 2b$$
$$1 \neq a \cdot 0 + b \cdot 0 = 0 \implies s_2 \notin M.$$

$$\implies S \neq M.$$

<u>Ex.</u>

$U \subset \mathbb{R}^4$

$$U: \begin{cases} 3x_2 + x_3 - x_4 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases}$$

$W \subset \mathbb{R}^4$ il più piccolo sottospazio che contiene

$$\underset{w_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} \text{ e } \begin{pmatrix} 0 \\ t \\ t+1 \\ t+2 \end{pmatrix}, \ t \in \mathbb{Z}$$

(a) dim e BASE DI U?

$$\begin{cases} x_1 = 3x_2 + x_3 \\ x_4 = -x_2 + 2x_3 \end{cases}$$

$$\begin{cases} -x_2 + 2x_3 = 3x_2 + x_3 \\ x_4 = -x_2 + 2x_3 \end{cases}$$

$$\begin{cases} x_3 = 4x_2 \\ x_4 = 7x_2 \end{cases} \Longrightarrow \dim U = 2$$

$$\text{BASE DI } U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ 7 \end{pmatrix} \right\}.$$

(b) eq. cartesiane + BASE DI W?

$$\begin{pmatrix} 0 \\ t \\ t+1 \\ t+2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{w_2} + t \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{w_3}$$

$$W = \langle w_1, w_2, w_3 \rangle$$

$$a w_1 + b w_2 + c w_3 = 0$$

$$\begin{pmatrix} a \\ a \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \\ 2b \end{pmatrix} + \begin{pmatrix} 0 \\ c \\ c \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} a = 0 \\ a + c = 0 \rightarrow c = 0 \\ b = 0 \end{cases} \Longrightarrow \text{LIN. IND.}$$

$$\begin{cases} x_1 = a \\ x_2 = a+c \\ x_3 = b+c \\ x_4 = 2b+c \end{cases}$$

$a = x_1$

$c = x_2 - x_1$

$b = x_3 - x_2 + x_1$

$x_4 = 2x_3 - 2x_2 + 2x_1 + x_2 - x_1$

$\boxed{x_1 - x_2 + 2x_3 - x_4 = 0}$: W

(c) $U \cap W$, $U + W$ ← BAf1 ?

$$U \cap W : \begin{cases} 3x_2 + x_3 - x_4 = 0 \\ x_2 - 2x_3 + x_4 = 0 \\ x_1 - x_2 + 2x_3 - x_4 = 0 \end{cases} \quad \begin{matrix} \} U \\ \\ \} W \end{matrix}$$

$$\begin{cases} x_4 = 3x_2 + x_3 = 7x_2 \\ x_2 - 2x_3 + 3x_2 + x_3 = 0, \quad 4x_2 = x_3 \\ x_1 - \cancel{x_2} + 8\cancel{x_2} - 7\cancel{x_2} = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 \\ x_3 = 4x_2 \\ x_4 = 7x_2 \end{cases} \Rightarrow \dim(U \cap W) = 1$$

BASE DI $(U \cap W) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 4 \\ 7 \end{pmatrix} \right\}$.

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$\overset{!}{=} 2 + 3 - 1 = 4$

$\Rightarrow U + W = \mathbb{R}^4$

BASE DI $(U+W) = \{e_1, e_2, e_3, e_4\}$.

(d) $L \subset \mathbb{R}^4$

$L \oplus U$
$L \oplus W$ $\exists$? se sì, BASE?

$\dim U = 2$

$\dim W = 3$

$\Rightarrow \dim L = 1$

$L = \langle \ell \rangle$ t.c. $U \cap L = \{\underline{0}\}$

$W \cap L = \{\underline{0}\}$

$\ell = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_4$