TUTORATO 23/03/22.

Ex. 1

(a) $S = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

$$v_1 = \begin{pmatrix} a_1 \\ b_1 \\ a_1 + b_1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} a_2 \\ b_2 \\ a_2 + b_2 \end{pmatrix}$$

$$\lambda_1 v_1 + \lambda_2 v_2 \overset{?}{\in} S$$

$$\begin{pmatrix} \lambda_1 a_1 + \lambda_2 a_2 \\ \lambda_1 b_1 + \lambda_2 b_2 \\ \lambda_1 a_1 + \lambda_1 b_1 + \lambda_2 a_2 + \lambda_2 b_2 \end{pmatrix} \in S$$

$\Longrightarrow$ S e' SOTTOSPAZIO.

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle \Longrightarrow \dim S = 2$$

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathcal{M}_{3,1}(\mathbb{R}) \;\middle|\; 2x + z = 0 \right\}$

$z = -2x$

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ -2x_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ -2x_2 \end{pmatrix}$$

$$\lambda_1 v_1 + \lambda_2 v_2 = \begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ -2\lambda_1 x_1 - 2\lambda x_2 \end{pmatrix} \in S$$

$\Longrightarrow$ S e' SOTTOSPAZIO.

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \Longrightarrow \dim S = 2.$$

Ex. 2

(a) $W = \{ p(x) \in \mathbb{R}^{\leq 3}[x] \mid p(0) = 0 \}$

$p(x) = ax^3 + bx^2 + cx + d$

$p_1(x) = a_1 x^3 + b_1 x^2 + c_1 x$

$p_2(x) = a_2 x^3 + b_2 x^2 + c_2 x$

$\lambda_1 p_1(x) + \lambda_2 p_2(x) \overset{?}{\in} W \Rightarrow$ sì

$\Rightarrow W$ e' SOTTOSPAZIO.

$W = \langle x^3, x^2, x \rangle \Rightarrow \dim W = 3$.

(c) $W = \{ p(x) \in \mathbb{R}^{\leq 3}[x]$ della forma $ax + bx^3$ dove $a, b \in \mathbb{R} \}$

$p_1(x) = a_1 x + b_1 x^3$

$p_2(x) = a_2 x + b_2 x^3$

$\lambda_1 p_1(x) + \lambda_2 p_2(x) =$

$= \lambda_1 a_1 x + \lambda_1 b_1 x^3 + \lambda_1 a_2 x + \lambda_2 b_2 x^3 =$

$= (\lambda_1 a_1 + \lambda_2 a_2) x + (\lambda_1 b_1 + \lambda_2 b_2) x^3$

$\Rightarrow W$ e' sottospazio

$W = \langle x^3, x \rangle \Rightarrow \dim W = 2$.

Ex. 3

(a) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \\ a+b & a-b \end{pmatrix} \in M_{3,2}(\mathbb{R}) \mid a,b \in \mathbb{R} \right\}$

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \\ a_1+b_1 & a_1-b_1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \\ a_2+b_2 & a_2-b_2 \end{pmatrix}$$

$$\lambda_1 A_1 + \lambda_2 A_2 = \begin{pmatrix} \lambda_1 a_1 + \lambda_2 a_2 & 0 \\ 0 & \lambda_1 b_1 + \lambda_2 b_2 \\ \lambda_1 a_1 + \lambda b_1 + \lambda_2 a_2 + \lambda_2 b_2 & \lambda_1 a_1 - \lambda_1 b_1 + \lambda_2 a_2 - \lambda_2 b_2 \end{pmatrix}$$

$\in S$

$\Longrightarrow S$ è SOTTOSPAZIO

$$S = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \right\rangle \Rightarrow \dim S = 2.$$

(c) $S = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in M_{2,3}(\mathbb{R}) \;\middle|\; a_{11} - a_{12} = 0 \right\}$

$a_{11} - a_{12} = 0 \iff a_{12} = a_{11}$

$A = \begin{pmatrix} a_{11} & a_{11} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{11} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

$\alpha A + \beta B = \begin{pmatrix} \alpha a_{11} + \beta b_{11} & \alpha a_{11} + \beta b_{11} & \alpha a_{13} + \beta b_{13} \\ \alpha a_{21} + \beta b_{21} & \alpha a_{22} + \beta b_{22} & \alpha a_{23} + \beta b_{23} \end{pmatrix}$

$\in S$

$\Longrightarrow S$ è SOTTOSPAZIO

$S = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \Rightarrow \dim S = 5.$

Ex. 4

(a) $W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; 2x+y+z=0 \text{ e } y-z=0 \right\}$

$W_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; 2x+y=0 \text{ e } y=0 \right\}$

BASE per W_1:

$$\begin{cases} z = y & \longrightarrow z = -x \\ 2x+y+y=0 & \longrightarrow y = -x \end{cases}$$

$$W_1 = \left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad \dim W_1 = 2$$

BASE per W_2:

$$\begin{cases} 2x+y=0 & \longrightarrow x = 0 \\ y=0 \end{cases}$$

$$W_2 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad \dim W_2 = 2$$

BASE di $W_1 \cap W_2$:

$$\begin{cases} \alpha_1 = 0 & \rightarrow \alpha_1 = 0 \\ -\alpha_1 = 0 \\ -\alpha_1 = \beta_1 & \rightarrow \beta_1 = 0 \\ \alpha_2 = \beta_2 \end{cases}$$

$$W_1 \cap W_2 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim(W_1 \cap W_2) = 1$$

BASE di $W_1 + W_2$:

$$W_1 + W_2 = \left\langle \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\begin{cases} a = 0 \\ -a = 0 \\ -a + c = 0 \\ b = 0 \end{cases} \longrightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} \Rightarrow \dim(W_1 + W_2) = 3.$$

(c) $W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; 2x + y + z = 0 \text{ e } y - z = 0 \right\}$

$$W_2 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \quad w$$

← BASE $\Rightarrow \dim W_2 = 1$

BASE DI W_1:

$$W_1 = \left\langle \underset{w_1}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\rangle \Rightarrow \dim W_1 = 2.$$

$$\alpha_1 w_1 + \alpha w_2 = \beta_1 w$$

$$W_1 \cap W_2 = ?$$

$$\begin{cases} \alpha_1 = \beta_1 \\ -\alpha_1 = \beta_1 \\ -\alpha_1 = \beta_1 \\ \alpha_2 = \beta_1 \end{cases} \rightarrow \begin{matrix} \alpha_1 = 0 \\ \beta_1 = 0 \\ \alpha_2 = 0 \end{matrix}$$

$$\Rightarrow W_1 \cap W_2 = \{\underline{0}\} \qquad \dim(W_1 \cap W_2) = 0$$

$$W_1 + W_2 = ?$$

$$W_1 + W_2 = \langle \overbrace{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}^{\text{BASE}} \rangle$$

$$\begin{cases} a + c = 0 \\ -a + c = 0 \\ -a + c = 0 \\ b + c = 0 \end{cases} \quad \begin{cases} c = -a \\ c = a \\ b = -c \end{cases} \rightarrow \begin{cases} a = 0 \\ c = 0 \\ b = 0 \end{cases}$$

$$\Rightarrow \dim(W_1 + W_2) = 3.$$

Ex. 5

(b) $W_1 = \{aX^3 + bX^2 + cX \mid a, b, c \in \mathbb{R}\}$

$W_2 = \{\alpha X^3 + \beta X^2 \mid \alpha, \beta \in \mathbb{R}\}$

$W_1 = \langle X^3, X^2, X \rangle \Rightarrow \dim W_1 = 3$

$W_2 = \langle X^3, X^2 \rangle \Rightarrow \dim W_2 = 2$

$aX^3 + bX^2 + cX = \alpha X^3 + \beta X^2$

$(a-\alpha)X^3 + (b-\beta)X^2 + cX = 0$

$c = 0$
$b = \beta$ $\quad W_1 \cap W_2 = \langle X^3, X^2 \rangle = W_2$
$a = \alpha$ $\quad \Rightarrow \dim(W_1 \cap W_2) = 2$

$W_1 + W_2 = \langle X^3, X^2, X \rangle = W_1$

$\Rightarrow \dim(W_1 + W_2) = 3$.

Ex. 6

(b) $W_1 = \left\{ \begin{pmatrix} a & a+b \\ 2a & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$

$W_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{11} + a_{12} = 0 \text{ e } a_{21} = 0 \right\}$

$\hookrightarrow a_{12} = -a_{11}$

$$W_1 = \Big\langle \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{BASE}} \Big\rangle \Rightarrow \dim W_1 = 2$$

$$W_2 = \Big\langle \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{BASE}} \Big\rangle \Rightarrow \dim W_2 = 2$$

$$a \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = c \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & a+b \\ 2a & 0 \end{pmatrix} = \begin{pmatrix} c & -c \\ 0 & d \end{pmatrix}$$

$$\begin{cases} a = c \longrightarrow c = 0 \\ a+b = -c \longrightarrow b = 0 \\ 2a = 0 \longrightarrow a = 0 \\ d = 0 \longrightarrow d = 0 \end{cases}$$

$W_1 \cap W_2 = \{ \underline{0}_{2\times 2} \}$

$\Rightarrow \dim(W_1 \cap W_2) = 0$

$$W_1 + W_2 = \left\langle \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{BASE}} \right\rangle$$

$$\left.\begin{cases} a + c = 0 \longrightarrow c = 0 \\ a + b - c = 0 \longrightarrow b = 0 \\ 2a = 0 \longrightarrow a = 0 \\ d = 0 \longrightarrow d = 0 \end{cases}\right\} \Rightarrow \text{LIN. IND.}$$

$$\Rightarrow \dim(W_1 + W_2) = 4 .$$

Ex. 7.

(c) $\delta = \left\{ \underset{v}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{w}{\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}} \right\}$? in $\mathbb{R}^3$

$$\alpha v + \beta w = 0 \iff \alpha = \beta = 0$$

$$\begin{cases} \alpha + 5\beta = 0 \longrightarrow \alpha = 0 \\ \beta = 0 \\ \beta = 0 \end{cases} \Rightarrow v, w \text{ LIN. IND.}$$

$$t = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \delta + \{t\} = \text{BASE di } \mathbb{R}^3 .$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \text{BASE di } \mathbb{R}^3 .$$

$$\begin{cases} \alpha + 5\beta = 0 \longrightarrow \alpha = 0 \\ \beta + \gamma = 0 \longrightarrow \gamma = 0 \\ \beta = 0 \end{cases} \Rightarrow \text{LIN. IND.}$$

Ex. 8

$\mathbb{R}^4$

(a) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$
$\quad v_1 \quad v_2 \quad v_3 \quad v_4$

$a v_1 + b v_2 + c v_3 + d v_4 = 0$

$$\begin{cases} a + b + c + d = 0 \longrightarrow a = 0 \\ b + c + d = 0 \longrightarrow b = 0 \\ c + d = 0 \longrightarrow c = 0 \\ d = 0 \end{cases}$$

$\Rightarrow$ LIN. IND.

$\Rightarrow$ BASE di $\mathbb{R}^4 = \{ v_1, v_2, v_3, v_4 \}$

Ex. 9

(c) $S = \{ x, x^2 \}$

$a x + b x^2 = 0 \iff a = b = 0$

$p(x) = a x^2 + b x + c$

$S + \{1\} = \{ x^2, x, 1 \}$ BASE di $\mathbb{R}^{\leq 2}[x]$

Ex. 10

(b) $S = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\}$

$\qquad\qquad A \qquad\qquad B \qquad\qquad C$

$aA + bB + cC = 0$

$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & -c \end{pmatrix} = \underline{0}_{2\times 2}$

$\begin{cases} -a-b = 0 \longrightarrow a = 0 \\ b + c = 0 \longrightarrow b = 0 \\ 0 = 0 \\ -c = 0 \longrightarrow c = 0 \end{cases} \Longrightarrow$ LIN. IND.

$S + \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} =$ BASE di $M_{2,2}(\mathbb{R})$

(d) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\qquad\qquad A \qquad\qquad B \qquad\qquad C \qquad\qquad D$

$aA + bB + cC + dD = 0$

$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2c & 0 \\ c & 0 \end{pmatrix} + \begin{pmatrix} 2d & 0 \\ 0 & d \end{pmatrix} = \underline{0}_{2\times 2}$

$$\begin{cases} a+2b+2c+2d=0 \longrightarrow a=0 \\ b=0 \\ c=0 \\ d=0 \end{cases}$$

$\Longrightarrow A, B, C, D$ LIN. IND.

$\Longrightarrow$ BASE di $M_{2,2}(\mathbb{R})$.

Es. 12

$\{v_1, v_2, \dots, v_n\}$ = BASE di V. $\Rightarrow \dim V = n$

(a) $\{v_1, v_2+\alpha v_1, v_3, \dots, v_n\}$ BASE di V.

$\lambda_1 v_1 + \lambda_2(v_2+\alpha v_1) + \lambda_3 v_3 + \dots + \lambda_n v_n = 0$

$\Leftrightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0$?

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_2 \alpha v_1 + \lambda_3 v_3 + \dots + \lambda_n v_n = 0$

$(\lambda_1 - \lambda_2\alpha) v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \dots + \lambda_n v_n = 0$

$\lambda_1 - \lambda_2 \alpha = 0 \longrightarrow \lambda_1 = 0$

$\lambda_2 = \lambda_3 = \dots = \lambda_n = 0$

$\Longrightarrow v_1, v_2+\alpha v_1, v_3, \dots, v_n$ LIN. IND.

$\Longrightarrow$ BASE di V.