TUTORATO 16/03/2022.

SPAZI VETTORIALI.

Ex. 1.

$\mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} = u \in \mathbb{R}^2 \quad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 v + \lambda_2 w$$

$$= \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} x = \lambda_1 - \lambda_2 \\ y = 2\lambda_1 + \lambda_2 \end{cases} \quad \begin{array}{l} \lambda_2 = \lambda_1 - x \\ y = 2\lambda_1 + \lambda_1 - x \end{array}$$

$$\lambda_1 = \frac{x+y}{3}$$

$$\lambda_2 = \frac{-2x+y}{3}$$

$$u = \begin{pmatrix} x \\ y \end{pmatrix} = \alpha_1 v + \alpha_2 w$$

$$u = \beta_1 v + \beta_2 w$$

$$\alpha_1 v + \alpha_2 w = \beta_1 v + \beta_2 w$$

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} \alpha_1 - \alpha_2 = \beta_1 - \beta_2 \\ 2\alpha_1 + \alpha_2 = 2\beta_1 + \beta_2 \end{cases}$$

$$\begin{cases} \alpha_1 = \alpha_2 + \beta_1 - \beta_2 \\ 2\alpha_2 + \cancel{2\beta_1} - 2\beta_2 + \alpha_2 = \cancel{2}\beta_1 + \beta_2 \end{cases}$$

$$\begin{cases} 3\alpha_2 = 3\beta_2 \longrightarrow \alpha_2 = \beta_2 \\ \alpha_1 = \beta_1 \end{cases}$$

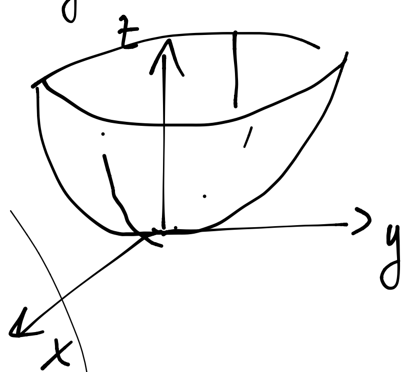
$$\iff \begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_2 \end{cases} \implies$$ si scrive in modo unico come comb. lin. di v e w.

Ex. 2 $\mathbb{R}^3$

(a) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \underline{x^2 + y^2 = z} \right\}$

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ x_1^2 + y_1^2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ x_2^2 + y_2^2 \end{pmatrix}$$

$$\lambda_1 v_1 + \lambda_2 v_2 \overset{?}{\in} S$$



$$\begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ \lambda_1 x_1^2 + \lambda_1 y_1^2 + \lambda_2 x_2^2 + \lambda_2 y_2^2 \end{pmatrix}$$

$$\neq (\lambda_1 x_1 + \lambda_2 x_2)^2 + (\lambda_1 y_1 + \lambda_2 y_2)^2$$

$\Longrightarrow S$ NON è sottospazio.

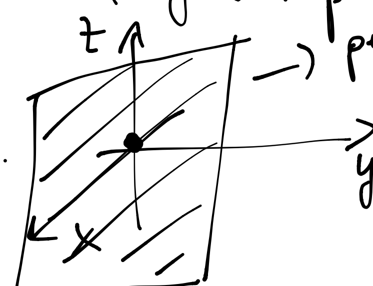
(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y = z \right\}$

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ x_1+y_1 \end{pmatrix} \quad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ x_2+y_2 \end{pmatrix}$$

$$\lambda_1 v_1 + \lambda_2 v_2 = \begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ \underline{\lambda_1 x_1} + \lambda_1 y_1 + \underline{\lambda_2 x_2} + \lambda_2 y_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ (\lambda_1 x_1 + \lambda_2 x_2) + (\lambda_1 y_1 + \lambda_2 y_2) \end{pmatrix}$$

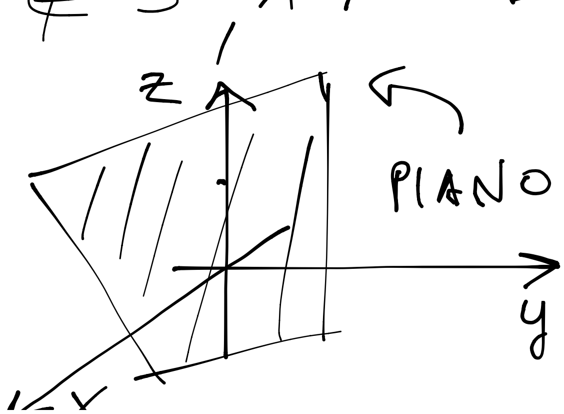
$\Longrightarrow S$ è sottospazio.



PIANO passante per $\underline{0}$

(e) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x+y-z+1=0 \right\}$

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S \quad 1 \neq 0 \Longrightarrow S$ NON è sottospazio.



PIANO $\underline{\text{NON}}$ passante per $\underline{0}$

Ex. 3.

(a) $\mathbb{R}^4$

$$V = \left\langle \underset{v_1}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}} \right\rangle \quad W = \left\langle \underset{w_1}{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}}, \underset{w_2}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}} \right\rangle$$

$$V+W = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle$$

$$w_1 = v_1 + v_2 - w_2$$

$$= \langle v_1, v_2, w_2 \rangle$$

$$V \cap W = ?$$

$$\alpha_1 v_1 + \alpha_2 v_2 = \beta_1 w_1 + \beta_2 w_2$$

$$\alpha_1 \begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \beta_1 \begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \beta_2 \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\begin{cases} \alpha_1 = \beta_2 \\ \alpha_2 = \beta_1 \\ 0 = 0 \\ \alpha_1 = \beta_1 \end{cases} \qquad \begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_1 \\ \beta_2 = \beta_1 \\ \beta_1 \text{ qualquer} \end{cases}$$

$$V \cap W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Ex. 5.

$\mathbb{R}^4$

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \underset{z=-x}{x+z=0}, \underset{w=3y}{3y-w=0} \right\}$$

$$T = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \underset{z=-x}{x+z=0}, \underset{y=-2w}{y+2w=0} \right\}$$

$$s_1 = \begin{pmatrix} x_1 \\ y_1 \\ -x_1 \\ 3y_1 \end{pmatrix} \in S, \quad s_2 = \begin{pmatrix} x_2 \\ y_2 \\ -x_2 \\ 3y_2 \end{pmatrix} \in S$$

$$\lambda_1 s_1 + \lambda_2 s_2 =$$

$$= \begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ -\lambda_1 x_1 - \lambda_2 x_2 \\ 3\lambda_1 y_1 + 3\lambda_2 y_2 \end{pmatrix} \begin{array}{l} \\ \\ \longleftarrow t = -x \\ \longleftarrow w = 3y \end{array}$$

$\Longrightarrow$ S è sottospazio.

$$t_1 = \begin{pmatrix} x_1 \\ -2w_1 \\ -x_1 \\ w_1 \end{pmatrix} \in T \qquad t_2 = \begin{pmatrix} x_2 \\ -2w_2 \\ -x_2 \\ w_2 \end{pmatrix} \in T$$

$$\lambda_1 t_1 + \lambda_2 t_2 \overset{?}{\in} T$$

$$\begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ -2\lambda_1 w_1 - 2\lambda_2 w_2 \\ -\lambda_1 x_1 - \lambda_2 x_2 \\ \lambda_1 w_1 + \lambda_2 w_2 \end{pmatrix} \in T$$

$\Longrightarrow$ T è sottospazio.

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\rangle \quad T = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = -2\beta_2 \\ -\alpha_1 = -\beta_1 \\ 3\alpha_2 = \beta_2 \end{cases} \qquad \begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = -2\beta_2 \\ \alpha_2 = \dfrac{\beta_2}{3} \\ \beta_1 \text{ qualsiasi} \end{cases} \longrightarrow \begin{cases} \beta_2 = 0 \\ \alpha_2 = 0 \end{cases}$$

Ad es. $\beta_1 = 1 \longrightarrow \alpha_1 = 1$

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow S \cap T = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

Ex.

a) S insieme delle funzioni reali definite nell'intervallo $[0,1]$, continue, positive o nulle.

f continua, positiva

$\Rightarrow -f$ continua, negativa $\notin S$

$\Rightarrow S$ NON è sottospazio.

(b) $S = \{ f : \mathbb{R} \to \mathbb{R},\ f \in C^2 \mid f'' + \omega^2 f = 0,\ \omega \in \mathbb{R} \}$.

$f, g \in S \overset{?}{\Rightarrow} f+g \in S$

$(f+g)'' + \omega^2 (f+g) = f'' + g'' + \omega^2 f + \omega^2 g =$

$= \underbrace{(f'' + \omega^2 f)}_{=0} + \underbrace{(g'' + \omega^2 g)}_{=0} = 0 + 0 = 0$

$$(\lambda f)'' + \omega^2 \lambda f = \lambda \, (\underbrace{f'' + \omega^2 f}_{=0}) = 0$$

$\Rightarrow S$ e' sottospazio.

(c) $S = \{$ funzioni reali $f(x)$, definite in $[0,1]$, continue $\mid \int_0^1 f(x) \sin(x)\, dx = 0 \}$.

$$f, g \in S \Rightarrow f+g \overset{?}{\in} S$$

$$\int_0^1 \underbrace{(f+g)(x)}\sin(x)\, dx =$$

$$= \underbrace{\int_0^1 f(x)\sin(x)\, dx}_{=0 \; f \in S} + \underbrace{\int_0^1 g(x) \sin(x)\, dx}_{=0 \; g \in S} = 0$$

$$\int_0^1 \lambda f(x) \sin(x)\, dx = \lambda \underbrace{\int_0^1 f(x) \sin(x)\, dx}_{=0} = 0$$

$\Rightarrow S$ e' sottospazio.

Ex.

$\mathbb{R}^4$

$$U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\rangle \qquad V = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$U+V$ è diretta?

$w \in U \cap V \iff$

$$w = \overset{=1}{\lambda_1} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \overset{=-1}{\lambda_2} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \mu_1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ 2\lambda_2 = \mu_1 \\ \lambda_1 + 3\lambda_2 = \mu_1 \\ 4\lambda_2 = \mu_1 + \mu_2 \end{cases} \qquad \begin{cases} \lambda_2 = -\lambda_1 \\ \mu_1 = -2\lambda_1 \\ -2\lambda_1 = -2\lambda_1 \to 0 = 0 \\ -4\lambda_1 = -2\lambda_1 + \mu_2 \end{cases}$$

$$\begin{cases} \lambda_2 = -\lambda_1 \\ \mu_1 = -2\lambda_1 \\ \mu_2 = -2\lambda_1 \\ \lambda_1 \text{ qls} \end{cases}$$

Scelgo $\lambda_1 = 1$.

$$U \cap V = \left\langle \begin{pmatrix} 0 \\ -2 \\ -2 \\ -4 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 2 \\ 2 \\ 4 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\rangle$$

$U \cap V \neq \{\underline{0}\} \Rightarrow U+V$ NON è diretta.

<u>Ex.</u>

$\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} ?$$

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \iff \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\begin{cases} \lambda_1 + 3\lambda_2 - \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 0 \\ \lambda_1 - \lambda_2 - \lambda_3 = 0 \end{cases} \qquad \begin{cases} \lambda_1 + 6\lambda_1 + \lambda_1 = 0 \\ \lambda_3 = -\lambda_1 \\ \lambda_2 = \lambda_1 + \lambda_1 = 2\lambda_1 \end{cases}$$

$$8\lambda_1 = 0 \Rightarrow \lambda_1 = 0$$

$$\Rightarrow \lambda_2 = 0, \ \lambda_3 = 0$$

$\Rightarrow v_1, v_2, v_3$ LIN. IND. $\Rightarrow$ BASE di $\mathbb{R}^3$.