

TUTORATO 09/03/2022.

## SISTEMI LINEARI.

Ex. 1.

$$\begin{cases} 2x_1 - 4x_2 = -4 \\ 3x_1 - 6x_2 + 3x_3 = -3 \\ x_1 - 2x_2 - x_3 = -2 \end{cases}$$

$$A|b = \left( \begin{array}{ccc|c} 2 & -4 & 0 & -4 \\ 3 & -6 & 3 & -3 \\ 1 & -2 & -1 & -2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ \textcircled{3} & -6 & 3 & -3 \\ 1 & -2 & -1 & -2 \end{array} \right) \quad \frac{I}{2}$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & -1 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} - 3\text{I} \\ \text{III} - \text{I} \end{array}$$

$$\rightarrow \left( \begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{array} \right) \begin{array}{l} \text{I} \\ \text{III} \\ \text{II} + 3\text{III} \end{array}$$

$0 = 3$  IMPOSS.  $\Downarrow$

$\Rightarrow$  NO SOLUTIONS!

Ex. 2.

$$\begin{cases} 2x_1 - 2x_2 + 8x_3 = 5 \\ 2x_2 + 6x_3 = 1 \\ x_1 - 2x_2 + 4x_3 = -1 \\ x_1 + 10x_3 = 0 \end{cases}$$

$\rightarrow A|b$

$$A|b = \left( \begin{array}{ccc|c} 2 & -2 & 8 & 5 \\ 0 & 2 & 6 & 1 \\ 1 & -2 & 4 & -1 \\ 1 & 0 & 10 & 0 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 2 & 6 & 1 \\ 2 & -2 & 8 & 5 \\ 1 & 0 & 10 & 0 \end{array} \right) \begin{array}{l} \text{III} \\ \text{II} \\ \text{I} \\ \text{IV} \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 2 & 6 & 1 \\ 0 & 2 & 0 & 7 \\ 0 & 2 & 6 & 1 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} - 2\text{I} \\ \text{IV} - \text{I} \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 2 & 0 & 7 \\ 0 & 2 & 6 & 1 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II}/2 \\ \text{III} \\ \text{IV} \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II}/2 \\ \text{III} - 2\text{II} \\ \text{IV} - 2\text{II} \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} / -6 \\ \text{IV} \end{array}$$

$$\begin{cases} x_1 - 2x_2 + 4x_3 = -1 \\ x_2 + 3x_3 = \frac{1}{2} \\ x_3 = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 = 3 \rightarrow x_1 = 3 + 7 = 10 \\ x_2 = \frac{1}{2} + 3 = \frac{7}{2} \\ x_3 = -1 \end{cases}$$

$$S = \left\{ x^* = \begin{pmatrix} 10 \\ 7/2 \\ -1 \end{pmatrix} \right\}.$$

Ex. 3.

$$\begin{cases} 2x_1 + 5x_3 = 1 \\ 4x_1 - 3x_2 + 4x_3 = 5 \\ 2x_1 - x_2 + 3x_3 = 2 \end{cases}$$

$$A|b = \left( \begin{array}{ccc|c} 2 & 0 & 5 & 1 \\ 4 & -3 & 4 & 5 \\ 2 & -1 & 3 & 2 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 4 & -3 & 4 & 5 \\ 2 & -1 & 3 & 2 \end{array} \right) \begin{array}{l} \text{I}/2 \\ \text{II} \\ \text{III} \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 0 & -3 & -6 & 3 \\ 0 & -1 & -2 & 1 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} - 4\text{I} \\ \text{III} - 2\text{I} \end{array}$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ 3\text{III} - \text{II} \end{array}$$

$$\left( \begin{array}{ccc|c} \textcircled{1} & 0 & 5/2 & 1/2 \\ 0 & \textcircled{1} & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} / -3 \\ \text{III} \end{array}$$

$$\begin{cases} X_1 + \frac{5}{2} X_3 = \frac{1}{2} \\ X_2 + 2X_3 = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 = -\frac{5}{2}x_3 + \frac{1}{2} \end{cases}$$

$$\begin{cases} x_2 = -2x_3 - 1 \end{cases}$$

$x_3$  qualsiasi

$$S = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2}x_3 + \frac{1}{2} \\ -2x_3 - 1 \\ x_3 \end{pmatrix} \mid x_3 \in \mathbb{R} \right\}$$

Ex. 4.

$m \in \mathbb{R}$

$$\begin{cases} mx + (m-1)y = m+2 \end{cases}$$

$$\begin{cases} (m+1)x - my = 5m+3 \end{cases}$$

$$A|b = \left( \begin{array}{cc|c} m & m-1 & m+2 \\ m+1 & -m & 5m+3 \end{array} \right)$$

•  $m = 0$

$$\left( \begin{array}{cc|c} 0 & -1 & 2 \\ 1 & 0 & 3 \end{array} \right) \rightarrow \begin{cases} x = 3 \\ y = -2 \end{cases}$$

•  $m \neq 0$

$$\left( \begin{array}{cc|c} m & m-1 & m+2 \\ m+1 & -m & 5m+3 \end{array} \right)$$

$$\left( \begin{array}{cc|c} 1 & \frac{m-1}{m} & \frac{m+2}{m} \\ m+1 & -m & 5m+3 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \end{array}$$

$$\left( \begin{array}{cc|c} 1 & \frac{m-1}{m} & \frac{m+2}{m} \\ 0 & \frac{1-2m^2}{m} & \frac{4m^2-2}{m} \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} - (m+1)\text{I} \end{array}$$

$$-m - (m+1) \frac{m-1}{m} = -m - \frac{m^2-1}{m} =$$

$$= \frac{1-2m^2}{m}$$

$$5m+3 - (m+1) \frac{(m+2)}{m} =$$

$$= \frac{5m^2 + 3m - m^2 - 3m - 2}{m} = \frac{4m^2 - 2}{m}$$

- $m^2 = \frac{1}{2} \Rightarrow m = \pm 1/\sqrt{2}$

$$\left( \begin{array}{c|c} \textcircled{1} & \frac{m-1}{m} \\ \hline 0 & 0 \end{array} \middle| \begin{array}{c} \frac{m+2}{m} \\ 0 \end{array} \right)$$

INFINITE SOLUTIONS

$$\left. \begin{array}{l} X = \frac{1-m}{m} y + \frac{m+2}{m} \\ y = \text{qualiasi} \end{array} \right\}$$

- $m^2 \neq \frac{1}{2}$

$$\left( \begin{array}{c|c} \textcircled{1} & \frac{m-1}{m} \\ \hline 0 & \frac{1-2m^2}{m} \end{array} \middle| \begin{array}{c} \frac{m+2}{m} \\ \frac{4m^2-2}{m} \end{array} \right)$$



$$y = \frac{4m^2 - 2}{\cancel{m}} \cdot \frac{\cancel{m}}{1 - 2m^2} = \frac{4m^2 - 2}{1 - 2m^2}$$

$$x = -\frac{m-1}{m}y + \frac{m+2}{m} = -2$$

$$= -\frac{m-1}{m} \cdot \frac{4m^2 - 2}{1 - 2m^2} + \frac{m+2}{m}$$

$$= \frac{2m - 2 + m + 2}{m} = \frac{3m}{m} = 3$$

$$\begin{cases} x = 3 \\ y = -2 \end{cases} \text{ UNICA SOL.}$$

EX. 5.

$$\lambda \in \mathbb{R}$$

$$\begin{cases} x + 2y + 2w + z = 1 \\ y + 2w + z = 0 \\ x + y + \lambda w = 0 \\ \lambda y + 2\lambda w + \lambda^2 z = 0 \end{cases}$$

$$A|b = \left( \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & \lambda & 0 & 0 \\ 0 & \lambda & 2\lambda & \lambda^2 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & \lambda-2 & -1 & -1 \\ 0 & \lambda & 2\lambda & \lambda^2 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} - \text{I} \\ \text{IV} \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda^2 - \lambda & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} + \text{II} \\ \text{IV} - \lambda \text{II} \end{array}$$

•  $\lambda = 0$

$\hookrightarrow$  NO sol.

•  $\lambda = 1$

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\begin{cases} x + 2y + 2w + z = 1 \\ y + 2w + z = 0 \\ w = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x + 2y + z = 3 \\ y + z = 2 \\ w = -1 \end{cases}$$

$$\begin{cases} w = -1 \\ y = 2 - z \\ x = 3 - z - 2y = 3 - z - 4 + 2z \\ \quad \quad \quad = z - 1 \\ z \text{ qualsiasi} \end{cases}$$

INFINITE sol.

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} z-1 \\ 2-z \\ -1 \\ z \end{pmatrix}, z \in \mathbb{R} \right\}$$

•  $\lambda \neq 0, 1$

$$\left( \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda^2 - \lambda & 0 \\ \lambda(\lambda-1) & & & & \end{array} \right)$$

$$z(\lambda^2 - \lambda) = 0 \rightarrow z = 0$$

$$\lambda w = -1 \rightarrow w = -1/\lambda$$

$$y = -2w - z = \frac{2}{\lambda}$$

$$x = -2y - 2w - z + 1$$

$$= -\frac{4}{\lambda} + \frac{2}{\lambda} + 1 = -\frac{2}{\lambda} + 1.$$

Ex. 6.

$$a \in \mathbb{R}$$

$$\begin{cases} x_1 - x_2 + ax_3 + 2x_4 = 0 \\ -x_1 + x_2 - 2ax_3 + ax_4 = 1 \\ 2x_2 + (a-2)x_3 + (1-2a)x_4 = -1 \\ x_1 - x_2 - ax_3 + 4x_4 = 2 \end{cases}$$

$$A|b = \left( \begin{array}{cccc|c} 1 & -1 & a & 2 & 0 \\ -1 & 1 & -2a & a & 1 \\ 0 & 2 & a-2 & 1-2a & -1 \\ 1 & -1 & -a & 4 & 2 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & -1 & a & 2 & 0 & \text{I} \\ 0 & 0 & -a & a+2 & 1 & \text{II} + \text{I} \\ 0 & 2 & a-2 & 1-2a & -1 & \text{III} \\ 0 & 0 & -2a & 2 & 2 & \text{IV} - \text{I} \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & -1 & a & 2 & 0 \\ 0 & 2 & a-2 & 1-2a & -1 \\ 0 & 0 & -a & a+2 & 1 \\ 0 & 0 & -2a & 2 & 2 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & -1 & a & 2 & 0 \\ 0 & 2 & a-2 & 1-2a & -1 \\ 0 & 0 & -a & a+2 & 1 \\ 0 & 0 & 0 & -2a-2 & 0 \end{array} \right) \begin{array}{l} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} - 2\text{III} \end{array}$$

•  $a \neq 0, -1 \quad -2a = 2 \Leftrightarrow a = -1$

$$(-2a-2)x_4 = 0 \rightarrow x_4 = 0$$

$$-ax_3 = 1 \rightarrow x_3 = -\frac{1}{a}$$

$$2x_2 = -(a-2)x_3 - 1$$

$$= \frac{1}{a} - 1$$

$$\rightarrow x_2 = \frac{a-2}{2a} - \frac{1}{2} = \frac{a-2-a}{2a} = -\frac{1}{a}$$

$$x_1 = x_2 - ax_3 = -\frac{1}{a} + a \frac{1}{a} = 1 - \frac{1}{a} = \frac{a-1}{a}$$

•  $\omega = -1$

$$\left( \begin{array}{cccc|c} 1 & -1 & -1 & 2 & 0 \\ 0 & 2 & -3 & 3 & -1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \leftarrow$$

$$\begin{cases} x_3 + x_4 = 1 \\ 2x_2 - 3x_3 + 3x_4 = -1 \\ x_1 - x_2 - x_3 + 2x_4 = 0 \\ 0 = 0 \end{cases}$$

$$x_3 = -x_4 + 1$$

$$x_2 = \frac{-3x_4 + 3 - 3x_4 - 1}{2}$$

$$= -3x_4 + 1$$

$$x_1 = -3x_4 + 1 - x_4 + 1 - 2x_4$$

$$= -6x_4 + 2$$

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6x_4 + 2 \\ -3x_4 + 1 \\ -x_4 + 1 \\ x_4 \end{pmatrix}, x_4 \in \mathbb{R} \right\}$$

•  $a = 0$

$$\left( \begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 2 & -2 & 1 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right)$$

$$\left( \begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 & \text{I} \\ 0 & 2 & -2 & 1 & -1 & \text{II} \\ 0 & 0 & 0 & 2 & 1 & \text{III} \\ 0 & 0 & 0 & 0 & 1 & \text{IV} + \text{III} \end{array} \right)$$

$\rightarrow 0 = 1$

$\Rightarrow$  NO SOL.