Esercizio 7.5.2 $x_1 \quad x_2 \quad x_3 \quad x_4$ variabili $a,b,c \in \mathbb{R}$ parametri

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & b \\ a & a & 0 & -2 & c \\ 0 & -a & 0 & a+1 & a \end{array}\right) \longrightarrow$$

$$\begin{array}{c} \mathrm{I} \\ \mathrm{II} - a\mathrm{I} \\ 2\,\mathrm{III} \end{array} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & b \\ 0 & 2a & 0 & -2(a+1) & c-ab \\ 0 & -2a & 0 & 2(a+1) & 2a \end{array}\right) \longrightarrow$$

$$\longrightarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}+\mathrm{II} \end{array} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & b \\ 0 & 2a & 0 & -2(a+1) & c-ab \\ 0 & 0 & 0 & 0 & 2a+c-ab \end{array}\right)$$

ha soluzioni se e solo se $2a - ab + c = 0$

cioè $a(b-2) = c$ ovvia $\boxed{ab = c+2a}$

In questo caso: $c - ab = -2a$ e il sistema

diventa:
$$\begin{cases} x_1 - x_2 + 2x_4 = b \\ \cancel{2}a\, x_2 - \cancel{2}(a+1)x_4 = -\cancel{2}a \end{cases}$$

se $a \neq 0$
$$\begin{cases} x_2 = \frac{a+1}{a} x_4 - a \\ x_1 = x_2 - 2x_4 + b = \frac{1-a}{a} x_4 + b \end{cases}$$

Soluzioni $S = \left\{ \left(\frac{1-a}{a} \cdot s + b,\ \frac{a+1}{a} s - a,\ t,\ s \right) \mid s,t \in \mathbb{R} \right\} =$

$$S_{a,b} = \left\{ (b, -a, 0, 0) + s\left(\frac{1-a}{a}, \frac{a+1}{a}, 0, 1\right) + t(0,0,1,0) \,/\, t,s \in \mathbb{R} \right\}$$

$$= (b, -a, 0, 0) + \left\langle \left(\frac{1-a}{a}, \frac{a+1}{a}, 0, 1\right), (0,0,1,0) \right\rangle$$

Se invece $a = 0$

$$\begin{cases} x_1 - x_2 + 2x_4 = b \\ x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = x_2 + b \\ x_4 = 0 \end{cases}$$

Sol. $S_{0,b} = \{ (t+b, t, s, 0) \,/\, s,t \in \mathbb{R} \} =$

$$= (b,0,0,0) + \langle (1,1,0,0), (0,0,1,0) \rangle$$

$\boxed{7.5.5}$ $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\mathcal{E}$ = base canonica

$$A = A_{\mathcal{E},\mathcal{E},L} = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 6 \\ 1 & 6 & 8 \end{pmatrix}$$

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ 3x+6z \\ x+6y+8z \end{pmatrix}$$

$$\mathrm{Ker}(L): \begin{cases} y+z=0 \\ 3x+6z=0 \\ x+6y+8z=0 \end{cases} \quad \xrightarrow{\mathrm{III}=6\mathrm{I}+\frac{1}{3}\mathrm{II}} \quad \begin{cases} y+z=0 \\ x+2z=0 \end{cases}$$

$$\Rightarrow \mathrm{Ker}(L) = \left\langle \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle \Rightarrow \dim \mathrm{Ker}(L) = 1$$

$$\Rightarrow \dim \mathrm{Im}(L) = 2 = \mathrm{rg}(A)$$

$$\mathrm{Im}(L) = \left\langle \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} \right\rangle$$

Domanda: determinare $L^{-1}((k,3,7))$ al variare di $k \in \mathbb{R}$.

$$\begin{cases} y+z=k \\ 3x+6z=3 \\ x+6y+8z=7 \end{cases} \qquad \left(\begin{array}{ccc|c} 0 & 1 & 1 & k \\ 3 & 0 & 6 & 3 \\ 1 & 6 & 8 & 7 \end{array}\right)$$

$$\begin{array}{c} \frac{1}{3}\mathrm{II} \\ \mathrm{I} \\ \mathrm{III}-\frac{1}{3}\mathrm{II} \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & k \\ 0 & 6 & 6 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & k \\ 0 & 0 & 0 & 6-6k \end{array}\right)$$

$\Rightarrow L^{-1}(k,3,7) \neq \emptyset$ se e solo se $k=1$.

In questo caso $(k=1)$

$$\begin{cases} x = 1 - 2z \\ y = 1 - z \end{cases}$$

Quindi, ponendo $z = t$ parametro otteniamo tutte le soluzioni:

$$S = \{(1-2t, 1-t, t) / t \in \mathbb{R}\} =$$

$$= (1,1,0) + \langle (-2,-1,1) \rangle$$