Ex da Libro 5.5.2

$$\mathbb{R}^4$$
$$\cup\!\!\mid$$

Sia $S = \langle (1,1,1,1), (1,0,-1,2), (-1,0,-1,1), (1,2,1,3) \rangle$

i) dim S

ii) trovare se esiste $T \subseteq \mathbb{R}^4$ sottosp. / $S \oplus T = \mathbb{R}^4$

iii) ——— $W \subseteq \mathbb{R}^4$ sottosp. / $W \neq \{0\}$, $W \oplus S \neq \mathbb{R}^4$

iv) ——— $V \subset \mathbb{R}^4$ sottosp. / $V + S = \mathbb{R}^4$, $V \cap S \neq \{0\}$

v) trovare basi e eq. cart. di S, T, W, V

$$\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \begin{cases} \alpha+\beta-\gamma+\delta=0 \\ \alpha+2\delta=0 \\ \alpha-\beta-\gamma+\delta=0 \\ \alpha+2\beta+\gamma+3\delta=0 \end{cases} \iff \begin{array}{l} \mathrm{II} \\ \mathrm{I}-\mathrm{III} \\ \mathrm{III} \\ \mathrm{IV} \end{array} \begin{cases} \alpha+2\delta=0 \\ 2\beta=0 \\ \alpha-\beta-\gamma+\delta=0 \\ \alpha+2\beta+\gamma+3\delta=0 \end{cases}$$

$$\iff \begin{array}{l} \mathrm{I} \\ \frac{1}{2}\mathrm{II} \\ \mathrm{III}-\mathrm{I}+\frac{1}{2}\mathrm{II} \\ \mathrm{IV} \end{array} \begin{cases} \alpha+2\delta=0 \\ \beta=0 \\ -\gamma-\delta=0 \\ \alpha+2\beta+\gamma+3\delta=0 \end{cases} \iff \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV}-\mathrm{I}-2\mathrm{II}+\mathrm{III} \end{array} \begin{cases} \alpha+2\delta=0 \\ \beta=0 \\ +\gamma+\delta=0 \\ 0=0 \end{cases}$$

esplicitiamo le variabili "a scala"

$$\begin{cases} \alpha=-2\delta \\ \beta=0 \\ \gamma=-\delta \end{cases}$$

cond. di dip. lineare $-2(1,1,1,1)+(-1)(-1,0,-1,1)+(1,2,1,3)=0$

$\Rightarrow (1,2,1,3)=2(1,1,1,1)+(-1,0,-1,1)$

$\Rightarrow S=\langle(1,1,1,1),(1,0,-1,2),(-1,0,-1,1)\rangle$

Continuiamo Ex 5.5.2:

Abbiamo $S = \langle (1,1,1,1), (1,0,-1,2), (-1,0,-1,1) \rangle$

Sono questi 3 lin. INDIP.? SI!

$\alpha(1,1,1,1) + \beta(1,0,-1,2) + \gamma(-1,0,-1,1) = (0,0,0,0)$

$\parallel$

$(\alpha+\beta-\gamma,\ \alpha,\ \alpha-\beta-\gamma,\ \alpha+2\beta+\gamma)$

$$\begin{cases} \alpha+\beta-\gamma = 0 \\ \alpha = 0 \\ \alpha-\beta-\gamma = 0 \\ \alpha+2\beta+\gamma = 0 \end{cases} \xrightarrow{I-II} \begin{cases} \alpha = 0 \\ 2\beta = 0 \\ \alpha-\beta-\gamma = 0 \\ \alpha+2\beta+\gamma = 0 \end{cases} \longrightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{cases}$$

Quindi $\dim S = 3$

Eq. Cartesiane per S?

$$\begin{cases} x_1 = \alpha+\beta-\gamma \\ x_2 = \alpha \\ x_3 = \alpha-\beta-\gamma \\ x_4 = \alpha+2\beta+\gamma \end{cases} \qquad \begin{cases} \alpha = x_2 \\ 2\beta = x_1 - x_3 \\ \gamma = \alpha-\beta-x_3 = x_2 - \frac{x_1}{2} + \frac{x_3}{2} - x_3 \\ x_4 = \alpha+2\beta+\gamma \end{cases}$$

$$\begin{cases} \alpha = x_2 \\ \beta = \frac{x_1-x_3}{2} \\ \gamma = x_2 - \frac{x_1}{2} - \frac{x_3}{2} \end{cases}$$

$$x_4 = x_2 + x_1 - x_3 + x_2 - \frac{x_1}{2} - \frac{x_3}{2} = \frac{1}{2}x_1 + 2x_2 - \frac{3}{2}x_3$$

Eq. Cart. $x_1 + 4x_2 - 3x_3 - 2x_4 = 0$

(ii) $\exists\ T$ t.c. $S \oplus T = \mathbb{R}^4$

l'eq. di S è $x_1 + 4x_2 - 3x_3 - 2x_4 = 0$

prendiamo $T = \langle (1,0,0,0) \rangle = \{ (t,0,0,0) \,/\, t \in \mathbb{R} \}$

allora $T \cap S = \{0\}$ e

$\{ (1,0,0,0), (1,1,1,1), (1,0,-1,2), (-1,0,-1,1) \}$ è lin. INDIP.

$\Rightarrow \quad T \oplus S = \mathbb{R}^4$

(iii) $\exists\ W \subseteq \mathbb{R}^4 \,/\, W \neq \{0\}$ e $W \oplus S \neq \mathbb{R}^4$?

NO: infatti $W \neq \{0\} \Rightarrow \dim W \geq 1$

e se $W \cap S = \{0\} \Rightarrow \dim (W \oplus S) = \dim S + \dim W \geq 3 + 1 = 4$

iv) $\exists\ V \leq \mathbb{R}^4$ sottosp. $/ \ V + S = \mathbb{R}^4$ e $V \cap S \neq \{0\}$

SI: prendiamo $V = \langle (1,0,0,0), (1,1,1,1) \rangle$

Allora $(1,1,1,1) \in V \cap S \neq \{0\}$

e $V + S = \langle (1,0,0,0), (1,1,1,1), (1,0,-1,+2), (-1,0,-1,1) \rangle = \mathbb{R}^4$

(iv) eq. cart. per T? $\begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$; per V? $\begin{cases} x_2 = x_3 \\ x_2 = x_4 \end{cases}$

Esercizio 5.5.3 : $\mathcal{M}_{2,2}(\mathbb{R})$

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \Big/ a, b \in \mathbb{R} \right\}$$

$$V = \left\{ \begin{pmatrix} d & 0 \\ d & d \end{pmatrix} \Big/ d \in \mathbb{R} \right\}$$

i) base di U e di V e eq. cartesiane

ii) base U+V e U∩V

iii) completare la base di U+V ad una base B di $\mathcal{M}_2(\mathbb{R})$

iv) coord. di $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ nella base B sopra.

i) $$U = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Big/ a, b \in \mathbb{R} \right\} =$$

$$= \underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle}_{\text{base}} \qquad \dim U = 2$$

eq. param. : $$\begin{cases} x_{11} = a \\ x_{12} = b \\ x_{21} = 0 \\ x_{22} = -a \end{cases} \iff \begin{cases} x_{21} = 0 \\ x_{22} + x_{11} = 0 \end{cases}$$

$$V = \left\{ \begin{pmatrix} d & 0 \\ d & d \end{pmatrix} / d \in \mathbb{R} \right\} = \left\{ d \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} / d \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

eq. cart.

$$\begin{cases} x_{11} = d \\ x_{12} = 0 \\ x_{21} = d \\ x_{22} = d \end{cases} \Longleftrightarrow \begin{cases} x_{11} = d \\ x_{12} = 0 \\ x_{21} = x_{11} \\ x_{22} = x_{11} \end{cases} \qquad \begin{cases} x_{12} = 0 \\ x_{21} - x_{11} = 0 \\ x_{22} - x_{11} = 0 \end{cases}$$

(ii) $$U \cap V = \left\{ \begin{pmatrix} d & 0 \\ d & d \end{pmatrix} \in V / \begin{matrix} d = 0 \\ d + d = 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\Longrightarrow \quad U + V = U \oplus V = \underbrace{\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle}_{\text{base}}$$

Eq. Cart: $$U + V = \left\{ \begin{pmatrix} a + c & b \\ c & -a + c \end{pmatrix} / a, b, c \in \mathbb{R} \right\} =$$

$$= \begin{cases} x_{11} = a + c \\ x_{12} = b \\ x_{21} = c \\ x_{22} = -a + c \end{cases} \qquad \begin{cases} a = x_{11} - x_{21} \\ b = x_{12} \\ c = x_{21} \\ x_{22} = -x_{11} + x_{21} + x_{21} \end{cases}$$

Eq. Cart. $x_{11} - 2x_{21} + x_{22} = 0$

(iii) Completamos la base con $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(iv) $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ base

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \delta \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha+\gamma+\delta & \beta \\ \delta & -\gamma+\delta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{cases} \alpha+\gamma+\delta = 1 \\ \beta = 1 \\ \delta = 1 \\ -\gamma+\delta = 1 \end{cases} \iff \begin{cases} \alpha = 0 \\ \beta = 1 \\ \delta = 1 \\ \gamma = 0 \end{cases}$$

In effetti $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

+ Intro :
- funzioni : in. suriett. biiett.
- appl. lineari

Es! V sp. vett. $v_1, \dots, v_k \in V$

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & V \\ \psi & & \psi \\ (x_1, \dots, x_k) & \longmapsto & x_1 v_1 + x_2 v_2 + \dots + x_k v_k \end{array}$$