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Cauchy – Riemann eqns

$f = u + iv$

$$\left(\begin{cases} u(x,y) = \operatorname{Re} f(x+iy) \\ v(x,y) = \operatorname{Im} f(x+iy) \end{cases}\right)$$

$f(z) = \underbrace{\operatorname{Re} f(z)}_{u(z)}$

f is $\mathbb{C}$ - diff at some $z = x+iy$

$\Leftrightarrow$

i) u, v are $\mathbb{R}$ - diff at (x,y)

ii) CR eqns hold at (x,y)

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

Proof: f $\mathbb{C}$dif at z $\Leftrightarrow$

$$\exists\ f'(z) = \lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h} - f'(z)\right) = 0$$

$$\Leftrightarrow \quad \lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

$$\Leftrightarrow \quad f(z+h) - f(z) - f'(z)h = o(h)$$

$$\Leftrightarrow \quad \boxed{f(z+h) = f(z) + f'(z)h + o(h)} \quad (\mathbb{C}\text{ - diff})$$

$$f'(z) \in \mathbb{C} = \boxed{\alpha + i\beta}$$

$$f = u + iv$$

$$z = \boxed{x + iy} \qquad z + h = \underbrace{(x+\xi)}_{} + i\,\underbrace{(y+\eta)}_{}$$

$$h = \boxed{\xi + i\eta}$$

$$o(h) = \underset{\in\mathbb{C}}{o_{\mathbb{C}}}(|h|) = o_{\mathbb{R}}\left(\sqrt{\xi^2+\eta^2}\right) + i\, o_{\mathbb{R}}\left(\sqrt{\xi^2+\eta^2}\right)$$

Plugging this into ($\mathbb{C}$-diff) we get

$$\overbrace{u(x+\xi,\, y+\eta) + i\, v(x+\xi,\, y+\eta)}^{f(z+h)} = \overbrace{u(x,y) + i\, v(x,y)}^{f(z)} + \overbrace{(\alpha + i\beta)\cdot(\xi + i\eta)}^{f'(z)\,\cdot\, h} + \overbrace{o_{\mathbb{R}}\left(\sqrt{\xi^2+\eta^2}\right) + i\, o_{\mathbb{R}}\left(\sqrt{\xi^2+\eta^2}\right)}^{o(h)}$$

$$(\alpha\xi - \beta\eta) + i\,(\beta\xi + \alpha\eta)$$

$$u((x,y)+(\xi,\eta)) = u(x,y) + \underbrace{(\alpha,-\beta)}_{}\cdot(\xi,\eta) + o(\|(\xi,\eta)\|)$$

$$\Leftrightarrow \begin{cases} u(x+\xi,\, y+\eta) = u(x,y) + (\alpha\xi - \beta\eta) + o_{\mathbb{R}}\left(\sqrt{\xi^2+\eta^2}\right) \\[2em] v(x+\xi,\, y+\eta) = v(x,y) + (\beta\xi + \alpha\eta) + o_{\mathbb{R}}\left(\sqrt{\xi^2+\eta^2}\right) \end{cases}$$

$$v((x,y)+(\xi,\eta)) = v(x,y) + \underbrace{(\beta,\alpha)}_{}\cdot(\xi,\eta) + o(\|(\xi,\eta)\|)$$

$\Leftrightarrow$ u is diff at (x,y)

and $\nabla u^{(x,y)} = (\alpha, -\beta)$

i) of statement

$\Leftrightarrow$ v is diff at (x,y)

and $\nabla v(x,y) = (\beta, \alpha)$

$$(\partial_x u, \partial_y u) = (\alpha, -\beta)$$

$$(\partial_x v, \partial_y v) = (\beta, \alpha)$$

$$\Rightarrow \begin{cases} \partial_x u = \alpha = \partial_y v \\ \partial_y u = -\beta = -\partial_x v \end{cases}$$

CR eqns.

▨

Examples & Applications of CR eqns

Ex: Let $f(z) := \operatorname{Im} z$, $z \in \mathbb{C}$.

Det at which z f is $\mathbb{C}$-diff.

Sol: Here

$$u + iv = f(x+iy) = y \Rightarrow \begin{matrix} u(x,y) = y \\ v(x,y) = 0 \end{matrix}$$

Now, f is $\mathbb{C}$ - diff at $z = x + iy$ $\iff$

i) u, v are $\mathbb{R}$-diff at (x,y)

True at every (x,y) because, according to diff test, $\partial_x u \equiv 0$, $\partial_y u \equiv 1$, $\partial_x v \equiv 0$, $\partial_y v \equiv 0 \in \mathcal{C}(\mathbb{R}^2)$

$\Downarrow$

u, v are $\mathbb{R}$-diff $\forall (x,y) \in \mathbb{R}^2$

ii) u, v fulfill CR eqns

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \quad \text{at } (x,y)$$

$$\Updownarrow$$

$$\begin{cases} 0 = 0 \\ 1 = -0 \quad \text{never} \end{cases}$$

$\Rightarrow$ i) & ii) toghether are never verified

$\Rightarrow \nexists f'(z) \quad \forall z \in \mathbb{C}.$ ■

Ex 6.10.5.

For each of the following $u(x,y)$ determine $v(x,y)$ s.t $f = u + iv$ be holomorphic on $\mathbb{C}$. Then det f as a funct of z

Def: f holomorphic on $D \subset \mathbb{C}$ means f $\mathbb{C}$ - diff $\forall z \in D$.

i) $u(x,y) = x$

Sol: $f = u + iv$ is $\mathbb{C}$ - diff $\forall \underset{\overset{\|}{x+iy}}{z} \in \mathbb{C}$

$\Longleftrightarrow$ i) u, v are $\mathbb{R}$ - diff at $\forall (x,y) \in \mathbb{R}^2$

ii) CR eqns hold $\forall (x,y) \in \mathbb{R}^2$, that is

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \quad \forall (x,y) \in \mathbb{R}$$

$\Longleftrightarrow$ (since u verifies i)

i) v is $\mathbb{R}$ - diff $\forall (x,y) \in \mathbb{R}^2$

ii) $$\begin{cases} 1 = \partial_y v \\ 0 = -\partial_x v \end{cases} \Longleftrightarrow \begin{cases} \boxed{\partial_x v = 0} \\ \partial_y v = 1 \end{cases} \forall (x,y) \in \mathbb{R}^2$$

$$\partial_x v = 0 \Longleftrightarrow v(x,y) = \varphi(y)$$

$$\partial_y v = 1 \Longleftrightarrow \partial_y v = \varphi'(y) = 1$$

$$\Longleftrightarrow \varphi(y) = y + c$$

$\Rightarrow v(x,y) = y + c$ ($c \in \mathbb{R}$ const)

And since this v verifies also i)

these v are the unique poss. functs for which

$$f = u+iv \quad \text{is} \quad \mathbb{C}\text{-diff on } \mathbb{C}.$$

$$f(\underbrace{x+iy}_{z}) = u(x,y) + iv(x,y)$$

$$= x + i(y+c)$$

$$= \underbrace{x+iy}_{z} + ic$$

$$\Rightarrow \quad f(z) = z + ic. \quad (c \in \mathbb{R}) \; ■$$

Rmk: So for inst

$u = \boxed{x}$

$v = 2y$ is $u+iv$ a $\mathbb{C}$-diff fuct ?

Here $f = u+iv = x+i2y$ and

$$\text{CR} \quad \left\{ \begin{array}{l} \partial_x u = \partial_y v \\ 1 = 2 \end{array} \right. \qquad \text{No!}$$

So f is not $\mathbb{C}$-diff at every $z \in \mathbb{C}$ ■

iii) $u(x,y) = x^2$

Here v is s.t. $f = u + iv$ is $\mathbb{C}$-diff (holomorphic) on $\mathbb{C}$ $\Leftrightarrow$

i) v is $\mathbb{R}$-diff on $\mathbb{R}^2$

(u is clearly $\mathbb{R}$-diff on $\mathbb{R}^2$)

ii) CR eqns hold

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \Leftrightarrow \begin{cases} 2x = \partial_y v \\ 0 = -\partial_x v . \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_x v = 0 \\ \partial_y v = 2x \end{cases} \quad \forall (x,y) \in \mathbb{R}^2$$

$$\Rightarrow v(x,y) = \varphi(y) \quad \Rightarrow \quad \varphi'(y) = 2x$$

$$\Rightarrow \quad \underbrace{\varphi(y) = 2xy + c}$$

not correct because φ must be const in x.

$\Rightarrow$ there is no f such that $\mathrm{Re} f = u = x^2$ and f is holomphic on $\mathbb{C}$. ∎

iv) $u(x,y) = x^2 - y^2$

Sol: $f = u + iv$ is holomorphic on $\mathbb{C}$ $\Leftrightarrow$

i) u, v are $\mathbb{R}$-diff $\forall (x,y) \in \mathbb{R}^2$

ii) CR eqns hold $\forall$ " "

i) u is clearly $\mathbb{R}$-diff on $\mathbb{R}^2$ (because $\partial_x u = 2x$, $\partial_y u = -2y \in \mathscr{C}(\mathbb{R}^2)$, diff. test applies)

So we look for a v diff on $\mathbb{R}^2$ s.t

$$\begin{cases} 2x = \partial_y v \\ -2y = -\partial_x v \end{cases} \quad \forall (x,y) \in \mathbb{R}^2$$

$$\Leftrightarrow \begin{cases} \partial_x v = 2y \Rightarrow v(x,y) = \int 2y \, dx + c(y) \\ \partial_y v = 2x \end{cases}$$

$$= 2yx + c(y)$$

$$\Updownarrow$$

$$\cancel{2x} + c'(y) = \cancel{2x} \; 0$$

$$\Leftrightarrow c'(y) = 0 \Rightarrow c(y) = c.$$

$$\Rightarrow \boxed{v(x,y) = 2xy + c.} \quad c \in \mathbb{R}$$

What is f in this case?

$$f = u + iv = x^2 - y^2 + i(2xy + c)$$

$$= \boxed{x^2 - y^2 + i\,2xy} + ic$$

$$f(x+iy) = (x+iy)^2 + ic \Rightarrow f(z) = z^2 + ic.$$

$c \in \mathbb{R}.$

Do 6.10.5. / 6.10.6.

Ex 4 (from Exams set)

Let

$$v(x,y) := e^{-y}(y\cos x + x\sin x)$$

i) Det all poss $u = u(x,y)$ s.t. $f = u + iv$ be $\mathbb{C}$ - diff on $\mathbb{C}$.

ii) Express $f = f(z)$.

Sol: $f = u+iv$ is $\mathbb{C}$ - diff on $\mathbb{C}$ $\Leftrightarrow$

i) u, v are $\mathbb{R}$ - diff on $\mathbb{R}^2$

ii) u, v verify CR eqns on $\mathbb{R}^2$.

i) About v, which is given, we have

$$\partial_x v = e^{-y}(-y\sin x + \sin x + x\cos x)$$

$$\partial_y v = e^{-y}\left[-(y\cos x + x\sin x) + (\cos x)\right]$$

Clearly $\partial_x v, \partial_y v \in \mathcal{C}(\mathbb{R}^2) \Rightarrow v$ is $\mathbb{R}$-diff on $\mathbb{R}^2$

So we look for u $\mathbb{R}$-diff on $\mathbb{R}^2$ s.t

$$\begin{cases} \partial_x u = \partial_y v = e^{-y}\left(-y\,\underline{\cos x} - \underline{x\sin x} + \cos x\right) \\ \boxed{\partial_y u} = -\partial_x v = \underline{e^{-y}\left(y\sin x - \sin x - x\cos x\right)} \end{cases}$$

$$u(x,y) = \int \qquad dx + c(y)$$

$$= -ye^{-y}\int \cos x\,dx + e^{-y}\int x\overset{\overset{(\cos x)'}{/\!/}}{\sin x}\,dx$$

$$+ e^{-y}\int \cos x\,dx + c(y)$$

$$= -ye^{-y}\sin x + \underline{e^{-y}}\left[x\cos x \underline{- \int \cos x\,dx}\right]$$

$$+ \underline{e^{-y}\int \cos x\,dy} + c(y)$$

$$u(x,y) = -ye^{-y}\sin x + e^{-y}x\cos x + c(y)$$

$$\Downarrow$$

$$\partial_y u = e^{-y}(y-1)\sin x - e^{-y}x\cos x + c'(y)$$

$$= y e^{-y}\sin x - e^{-y}(x\cos x + \sin x) + c'(y)$$

$$\partial_y u = e^{-y}(y\sin x - x\cos x - \sin x) + c'(y)$$

$$\overset{?}{=} -\partial_x v = e^{-y}(y\sin x - \sin x - x\cos x)$$

$$\iff \quad c'(y) = 0 \quad \Rightarrow \quad c(y) = c$$

$$u(x,y) = e^{-y}(-y\sin x + x\cos x) + c.$$

ii) $f = u + iv$

$$= \underbrace{e^{-y}}(-y\sin x + x\cos x) + c +$$

$$i\,\underbrace{e^{-y}}(y\cos x + x\sin x)$$

$f(z)$ $\quad z = x + iy$

$$= e^{-y}\Big[\, y(-\underbrace{\sin x + i\cos x}) + x\big(\overset{\boxed{e^{ix}}}{\overset{\|}{\cos x + i\sin x}}\big)\Big]$$

$$i(\cos x + i\sin x)$$

$$\boxed{e^{ix}}$$

$$= e^{-y}\left[\, iy\, e^{ix} + x\, e^{ix} \right] = e^{ix-y}(iy+x)$$

$$= e^{i(x+iy)}(x+iy) = ze^{iz}. \qquad \blacksquare$$

⌜ $f(z) = e^{z}(iz+3) = \qquad u = \mathrm{Re} f$ ⌟

Ex 10

What are CR eqns?

Pb: Let $f = u + iv$ $(u = \mathrm{Re} f,\ v = \mathrm{Im} f)$ be $\mathbb{C}$-diff on $\mathbb{C}$.

Assume that also $\bar{f} = \underline{u - iv}$ is $\mathbb{C}$-diff on $\mathbb{C}$

$$= \boxed{u} + i\underbrace{(-v)}$$

What can be said on f?

Sol: By first assumption

i) u, v are $\mathbb{R}$-diff on $\mathbb{R}^2$

ii) u, v fulfill CR eqs on $\mathbb{R}^2$

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

By second ass.

i) $u, -v$ are $\mathbb{R}$-diff on $\mathbb{R}^2$

ii) CR eqns hold

$$\begin{cases} \partial_x u = \partial_y(-v) = -\partial_y v \\ \partial_y u = -\partial_x(-v) = +\partial_x v \end{cases}$$

$$2\partial_x u = \partial_y v - \partial_y v = 0$$

$$\partial_y u = 0$$

$$\Leftrightarrow \quad \begin{cases} \partial_x u = 0 \\ \partial_y u = 0 \end{cases} \qquad \begin{cases} \partial_x v = 0 \\ \partial_y v = 0 \end{cases}$$

$$u(x,y) = c(\cancel{y}) \qquad v(x,y) = d$$

$$\Leftrightarrow$$

$$\Rightarrow \quad f = u + iv = c + id \quad \text{constant.} \quad \blacksquare$$

Do Ex 14, 19, 25. (exam set).