Monday, 15 January 2024 08:39

Es. 6.10.2

$\# \qquad \mathrm{Ch}^2 z + 1 = 0$

$$\iff \quad \mathrm{Ch}^2 z = -1$$

$$\iff \quad \mathrm{ch}\, z = +i \quad , \quad \mathrm{Ch}\, z = -i$$

$$\shortparallel$$

$$\frac{e^z + e^{-z}}{\cancel{2}} = 2i$$

$$\Updownarrow \quad e^z \neq 0$$

$$e^{2z} + 1 = 2i\, e^z$$

$$\Updownarrow$$

$$(e^z)^2 \underbrace{-2i}\, e^z + 1 = 0$$

$$\Updownarrow \qquad = \sqrt{-8} = i2\sqrt{2}$$

$$e^z = \frac{2i \pm \sqrt{-4-4}}{2}$$

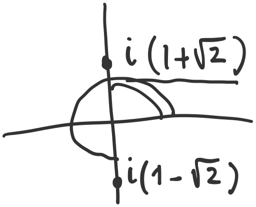
$$= \frac{\cancel{2}i \pm i\cancel{2}\sqrt{2}}{\cancel{2}}$$

$$e^{z} = i\,(1 \pm \sqrt{2})$$

$$\Updownarrow$$

$$z = \log\big(\underbrace{i\,(1 \pm \sqrt{2})}\big)$$

$$\ulcorner\ \log w = \log|w| + i\,(\arg w + k2\pi) \qquad k \in \mathbb{Z}\ \lrcorner$$



$$z_{+} = \log(1+\sqrt{2}) + i\left(\frac{\pi}{2} + k2\pi\right) \qquad k \in \mathbb{Z}$$

$$z_{-} = \log(\sqrt{2}-1) + i\left(\frac{3}{2}\pi + k2\pi\right)$$

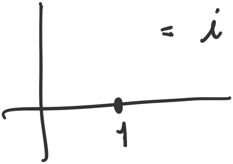
Sol of $\operatorname{Ch} z = -i$ is similar. ■

#6 $$e^{\boxed{z^2}} = 1$$

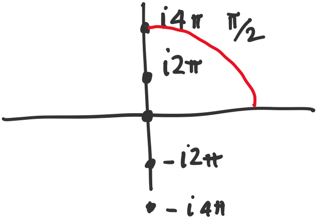
complex log ↙ $\qquad$ $\mathbb{R}$ log ↙

$$z^2 = \log 1 = \log|1| + i\,(0 + k2\pi)$$

$$= i\ k2\pi$$



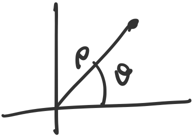
$$\Rightarrow \quad z^2 = \boxed{i\ k2\pi} \qquad k \in \mathbb{Z}$$



$$k = 0 \Rightarrow z = 0$$

$k > 0$:

$$z^2 = i\ k2\pi = k2\pi\ e^{i\boxed{\frac{\pi}{2}}}$$

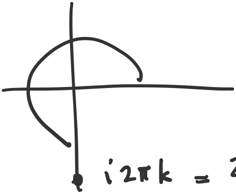


$$w = \rho e^{\boxed{i\theta}}$$

$$\Rightarrow \quad z = \sqrt{k2\pi}\ e^{i\left(\pi/4 + k\frac{2\pi}{2}\right)}$$

$$= \sqrt{2\pi k}\ e^{i\left(\frac{\pi}{4} + k\pi\right)} \qquad k>0$$

For $k<0$



$$i2\pi k = 2\pi|k|\ e^{i\boxed{\frac{3}{2}\pi}} = z^2$$

$$\Rightarrow \quad z = \sqrt{2\pi|k|}\; e^{i\left(\frac{3}{4}\pi + k\frac{\cancel{2}\pi}{\cancel{2}}\right)}$$

■

Rmk:

$$\boxed{\log(zw)} \overset{?}{=} \log z + \log w$$

$$\|$$

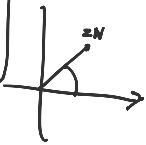
$$\boxed{\log|zw| + i\left(\underset{\in [0,2\pi[}{\boxed{\arg(zw)}} + \underbrace{k2\pi}\right)}$$

$$\log|zw| = \log(|z||w|)$$

$$\neq$$

$$\|$$

$$(\log|z| + \log|w|) + i\left((\arg z + \arg w) + k2\pi\right)$$

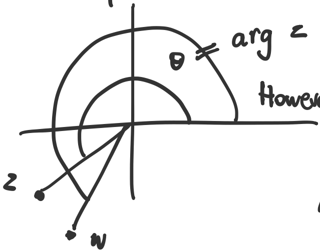


$$z = \rho e^{i\theta} \quad (\theta \leftarrow \arg z)$$

$$w = r e^{i\varphi} \quad (\varphi \leftarrow \arg w) \Rightarrow \boxed{zw = \rho r\, e^{i(\theta+\varphi)}}$$

$$\theta + \varphi = \arg(zw) + j2\pi$$

$$\varphi = \arg w \in [0, 2\pi[$$



$$\arg z \in [0, 2\pi[$$

However, $\boxed{\theta + \varphi} > 2\pi$

$$\neq$$

$$\boxed{\arg(zw)}$$

$$\Rightarrow \quad \log(zw) = \log|z| + \log|w| + i\left(\arg z + \arg w + \textcircled{k}2\pi\right)$$

$$= \underbrace{\log|z| + i\left(\arg z + \textcircled{k}2\pi\right)}_{\log z} + \underbrace{+ \log|w| + i\left(\arg w + \textcircled{h}2z\right)}_{\log w}$$

$$\Rightarrow \quad \log(zw) = \log z + \log w$$

However this relation is false for $\log_{\mathbb{C}}$

$$\log_{\mathbb{C}}(zw) = \log|zw| + i \underset{\in[0,2\pi]}{\arg(zw)}$$

$$= \log|z| + \log|w| + i\left(\overset{\#}{\arg z + \arg w}\right)$$

$$\neq \log_{\mathbb{C}} z + \log_{\mathbb{C}} w .$$

(this if $\arg z, \arg w > \pi$) ∎

Rmk: For $\mathbb{R}$ exp and log we have

$$e^{\log x} = x$$

$$\Rightarrow \quad x^{\alpha} = e^{\log x^{\alpha}} = e^{\alpha \log x}.$$

Def: $z, w \in \mathbb{C}$

$$z^{w} := e^{w \log z}. \qquad (*)$$

Ex: $$z^{1/2} = e^{\frac{1}{2}\log z} \underset{(*)}{} \overset{?}{=} \sqrt{z} \quad ?$$

If $$z = \rho e^{i\theta}$$

$$\sqrt{z} = \sqrt{\rho}\, e^{i\left(\frac{\theta}{2} + k\frac{2\pi}{2}\right)} = \sqrt{\rho}\, e^{i\left(\frac{\theta}{2} + k\pi\right)}$$

(De Moivre)

By using $(*)$

$$z^{1/2} = e^{\frac{1}{2}\left(\log|z| + i(\theta + k2\pi)\right)}$$

$$= e^{\underbrace{\log|z|^{1/2}}_{} + \underbrace{i\left(\frac{\theta}{2} + k\pi\right)}_{}}$$

$$= \underbrace{|z|}_{\rho}^{1/2} \; e^{i\left(\frac{\theta}{2}+k\pi\right)}$$

$$= \sqrt{\rho}\; e^{i\left(\frac{\theta}{2}+k\pi\right)}$$

■

Ex $$i^{i} = e^{i \log i}$$

$$\log i = \log 1 + i\left(\overset{\arg i}{\frac{\pi}{2}} + k2\pi\right)$$

$$= 0 + i\left(\frac{\pi}{2} + k2\pi\right)$$

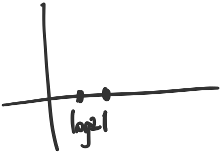
$$i \log i = i \cdot i \left(\frac{\pi}{2} + k2\pi\right) = -\left(\frac{\pi}{2} + k2\pi\right)$$

$$\Rightarrow \quad i^{i} = e^{-\left(\frac{\pi}{2} + k2\pi\right)} \qquad k \in \mathbb{Z}.$$

■

Ex: $1^{z} = 2.$

$$\boxed{1^{z}} = e^{z \log 1} = e^{z\, i\, 2k\pi}$$



$$\underset{\mathbb{C}\ \log}{\log 1} = \underset{\mathbb{R}\ \log}{\overset{0}{\overset{\parallel}{\log|1|}}} + i(0 + k2\pi)$$

$$= ik2\pi$$

$$1^z = 2 \iff e^{i\,2k\pi z} = 2$$

$$\iff i2\underset{\smile}{k}\pi z = \log_{\mathbb{C}} 2$$

$$= \log_{\mathbb{R}} 2 + i(0 + h2\pi)$$

$$\overset{k \neq 0}{\iff} \quad z = \frac{\log 2 + ih2\pi}{ik2\pi}$$

$$z = -i\frac{\log 2}{k2\pi} + \frac{h}{k}$$

(something wrong?)

∎

Let's return on $\mathbb{C}$-diff.

Prop: If f, g are $\mathbb{C}$-diff at z

$\Rightarrow$

- $\exists\, (f \pm g)'(z) = f'(z) \pm g'(z)$

- $\exists\, (\alpha f + \beta g)'(z) = \alpha f'(z) + \beta g'(z)$

$\alpha, \beta \in \mathbb{C}$

(linearity)

- $\exists\, (fg)'(z) = f'(z)\, g(z) + f(z)\, g'(z)$

- $\exists\, \left(\frac{f}{g}\right)'(z) = \dfrac{f'(z)\, g(z) - f(z) g'(z)}{g(z)^2}$

(provided $g(z) \neq 0$)

(Same proof as for the $\mathbb{R}$-derivative Calc)

Ex: $$\left(\sum_{n=0}^{N} c_n z^n\right)' = \left(c_0 + c_1 z + c_2 z^2 + \dots\ c_N z^N\right)'$$

$$= c_1 + 2c_2 z + 3c_3 z^2 + \dots + N c_N z^{N-1}$$

Prop: Power series $\sum_{n=0}^{\infty} c_n z^n$ are $\mathbb{C}$-diff in their disk of convergence and

$$\left(\sum_{n=0}^{\infty} c_n z^n\right)' = \sum_{n=0}^{\infty} (n+1)\, c_{n+1}\, z^n$$

$$\forall z : |z| < \underset{\substack{\uparrow \\ \text{radius of} \\ \text{conv.}}}{R}$$

Ex: Exp funct:

$$e^{z)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad z \in \mathbb{C}$$

$$(R = +\infty)$$

$$\Rightarrow \quad (e^z)' = \left(\sum_{n=0}^{\infty} \overset{c_n}{\boxed{\frac{1}{n!}}}\, z^n\right)'$$

$$= \sum_{n=0}^{\infty} \cancel{(n+1)} \underset{c_{n+1}}{\boxed{\frac{1}{(n+1)!}}}\, z^n \qquad (n+1)! = \cancel{(n+1)}\, n!$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

$$\Rightarrow \qquad (e^z)' = e^z \quad \forall z \in \mathbb{C}.$$

■

Ex: $\text{Sh}' z = \text{Ch } z \quad \forall z \in \mathbb{C}.$

We use the def .

$$\text{Sh } z = \frac{e^z - e^{-z}}{2}$$

$$(\text{Sh } z)' = \left(\frac{1}{2} e^z - \frac{1}{2} \frac{1}{e^z}\right)'$$

$$\overset{\text{lin}}{=} \frac{1}{2}(e^z)' - \frac{1}{2}\left(\frac{1}{e^z}\right)'$$

$$= \frac{1}{2} e^z + \frac{1}{2}\left(+ \frac{\cancel{e^z}\,1}{(e^z)^{\cancel{2}}}\right)$$

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

$$= \frac{1}{2}\left(e^z + e^{-z}\right) = \text{Ch } z.$$

■

Similarly $(\text{Ch } z)' = \text{Sh } z. \quad \forall z \in \mathbb{C}$

Ex: $(\sin z)' = \cos z \,, \quad (\cos z)' = -\sin z$

$\forall z \in \mathbb{C}.$

Here we use power series representation:

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad R = +\infty$$

$$\Rightarrow \exists \, (\sin z)' = \left(\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \right)'$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{\underset{\cancel{(2k+1)}(2k)!}{(2k+1)!}} \cancel{(2k+1)} \, z^{2k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = \cos z. \qquad \blacksquare$$

Pb: $\underbrace{(\log_{\mathbb{C}} z)'}_{\text{principal log}} \overset{?}{=} \frac{1}{z}$ (see later)

Ex:A $z \xrightarrow{f} \bar{z}$: f is never diff

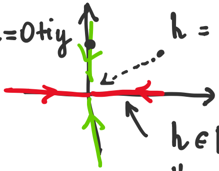
$x+iy \to \boxed{x}-\boxed{iy}$ $\qquad \nexists\, f'(z)\ \forall z \in \mathbb{C}.$

$$f'(z) \text{ exists} \Leftrightarrow \exists \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

Now

$$\frac{f(z+h)-f(z)}{h} = \frac{\overset{\cancel{\bar{z}}+\bar{h}}{\overset{\parallel}{\overline{z+h}}} - \cancel{\bar{z}}}{h} = \frac{\bar{h}}{h}$$

$h=0+iy$



$h = x+iy = (x,y)$ $\qquad h \to 0?$

$h \in \mathbb{R}$
$\parallel$
$x+i0 = x$ $\qquad \boxed{\frac{\bar{h}}{h}} = 1 \xrightarrow[h\to 0]{} 1$

$\bar{h} = x - i0 = x$ $\qquad \#$

$\bar{h} = 0 - iy$ $\qquad \frac{\bar{h}}{h} = \frac{-iy}{iy} = -1 \xrightarrow[h\to 0]{} -1$

$$\Rightarrow \nexists \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$\Rightarrow \nexists f'(z). \qquad \blacksquare$$

Ex: B $f(z) = \operatorname{Re} z \qquad \nexists f(z) \quad \forall z \in \mathbb{C}$

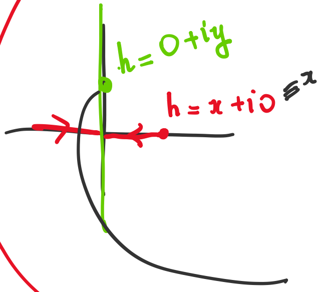
$$\left(f(x+iy) = x \right)$$

In this case

$$\cancel{\operatorname{Re} z} + \operatorname{Re} h$$
$$\|$$
$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{\operatorname{Re}(z+h) - \cancel{\operatorname{Re} z}}{h}$$

$$= \lim_{h\to 0} \frac{\operatorname{Re} h}{h}$$

$\nexists$



$$\frac{\operatorname{Re} h}{h} = \frac{x}{x} = 1 \xrightarrow[h\to 0]{} 1$$

$$\neq$$

$$\frac{\operatorname{Re} h}{h} = \frac{0}{iy} \equiv 0 \xrightarrow[h\to 0]{} 0$$

$\blacksquare$

To understand what is behind these last examples, let

$$f = \mathrm{Re} f + i \,\mathrm{Im} f$$

$$\overset{=}{f(x+iy)} = \boxed{\mathrm{Re} f(x+iy)} + i \boxed{\mathrm{Im} f(x+iy)}$$

$$\underset{u(x,y)}{\overset{\shortparallel}{}} \qquad \underset{v(x,y)}{\overset{\shortparallel}{}}$$

$$u, v : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Pb: under which conds on u, v is f $\mathbb{C}$- diff ?

Ex: Suppose that

$$\begin{cases} u(x,y) = x^2 + y^2 \\ v(x,y) = xy \end{cases} \Rightarrow \quad u+iv \text{ is } \mathbb{C}\text{-diff?}$$

In the case of

Ex A : $u(x,y) = x$

$v(x,y) = -y$

Ex B : $u(x,y) = x$

$v(x,y) = 0$

$\Big\}$ In both cases $u+iv$ is NOT $\mathbb{C}$-diff

Thm: Let $f = u+iv$. Then, f is $\mathbb{C}$-diff on D if and only if

i) u, v are $\mathbb{R}$-diff on D

ii) the following eqns must be verified

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

(Cauchy-Riemann eqns)

Ex: In case of Ex A

$u = x$

$v = -y$, clearly i) is verified but ii) is not because

$$\begin{cases} 1 = \partial_x u \overset{?}{=} \partial_y v = -1 & \text{never} \\ 0 = \partial_y u \overset{?}{=} -\partial_x v = 0 & \checkmark \end{cases}$$

CR conds failed.

For Ex B

$u = x$ $\quad$ u, v are $\mathbb{R}$-diff (i) true)

$v = 0$

$$\text{CR} \begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \Leftrightarrow \begin{cases} 1 = 0 \text{ never} \\ \end{cases}$$

(ii) false)

Ex: Assume

$$u = x^2 + y^2 , \quad v = xy$$

Is $u + iv$ $\mathbb{C}$-diff ?

Sol: $u + iv$ is $\mathbb{C}$ - diff $\Leftrightarrow$

i) u, v are $\mathbb{R}$ - diff (true)

ii) CR eqns hold

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \iff \begin{cases} 2x = x \\ 2y = -y \end{cases}$$

$$\iff (x,y) = (0,0)$$

$\Rightarrow$ $f = u + iv$ is $\mathbb{C}$ - diff at $(0,0)$ ☒