Thursday, 11 January 2024 08:32

Ex 6.0.1

Compute the radius of conv for

#1 $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$

Sol: Here $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} z^n$, $c_n = \frac{1}{n}$.

The radius of conv R is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1/n}{1/(n+1)} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{n} = 1. \qquad \blacksquare$$

#3 $\displaystyle\sum_{n=0}^{\infty} 2^n z^n = \sum_{n=0}^{\infty} (2z)^n$

Here $c_n = 2^n$. The radius of conv is

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \right| = \frac{1}{2}.$$

$\blacksquare$

#5. $\displaystyle \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n .$

Here $c_n = \dfrac{(n!)^2}{(2n)!}$ and

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} \cdot \frac{(\overset{2n+2}{\overset{\shortparallel}{2(n+1)}})!}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{n!}}{\underset{(n+1)\cancel{n!}}{(n+1)!}} \right)^2 \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4. \quad \blacksquare$$

Rmk: Let

$$f(z) := \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+i}.$$

This is a power series $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ where

$$c_n = \begin{cases} 0 & n \text{ even} \quad (n=2k) \\ \dfrac{1}{2k+i} & n \text{ odd} \quad (n=2k+1) \end{cases}$$

In part, in this case, we cannot compute

$$\left(\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+i} = \frac{z}{i} + \frac{z^3}{2+i} + \frac{z^5}{4+i} + \cdots \right.$$

$$\left. = 0\cdot z^0 + \frac{1}{i} z^1 + 0\cdot z^2 \cdots \right)$$

$$R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right|$$

because, for n odd, $\dfrac{c_n}{c_{n+1}} = \dfrac{1/2k+i}{0}$?

However, to discuss abs conv. of our series,

$$\sum_{k=0}^{\infty} \underbrace{\left| \frac{1}{2k+i} z^{2k+1} \right|}_{a_k}$$

Applying the ratio test to $\sum \overset{0}{\overset{1}{a}}_k$,

we have

$$\sum a_k \text{ converges if } q = \lim \frac{a_{k+1}}{a_k} < 1$$

diverges + $a_k \to \infty$

if $q > 1$

$$q = \lim \frac{a_{k+1}}{a_k} = \lim \frac{\left| \frac{1}{2(k+1)+i} \, z^{\cancel{2k}+3} \right|}{\left| \frac{1}{2k+i} \, z^{\cancel{2k}+1} \right|}$$

$$= \lim_{k \to \infty} \underbrace{\left| \frac{2k+i}{2k+2+i} \right|}_{=1} |z|^2$$

$|z|^{\cancel{2}} < 1 \Rightarrow \Sigma$ conv

$= |z|^2$ so $>1 \Rightarrow \Sigma$ div + $| \ | \to \infty$

$\blacksquare$

Exponential

Recall that

$$e^x = 1 + x + o(x)$$

$$= 1 + x + \frac{x^2}{2} + o(x^2)$$

$$+\frac{x^3}{3!}+o(x^3)$$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$$

Thm: $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \forall\, x \in \mathbb{R}.$$

Def:

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (\text{exp. series})$$

Prop: exp series has $R = +\infty$ (so, in part, e^z makes sense $\forall z \in \mathbb{C}$). Moreover

i) $e^z = e^x \quad \underset{x}{\underset{\shortparallel}{z}} \in \mathbb{R}$ (thm above)

ii) $e^0 = 1$

iii) $e^{z+w} = e^z e^w \quad \forall\, z, w \in \mathbb{C}$

iv) $\overline{e^z} = e^{\bar{z}} \quad \forall\, z \in \mathbb{C}.$

v) $e^z = e^x e^{iy} = e^x(\cos y + i \sin y) \quad z = x + iy$

Proof: The radius of conv of the exp series is

$$R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{1/n!}{1/(n+1)!} \right|$$

$$\left(c_n = 1/n! \right) \qquad = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \lim (n+1) = +\infty$$

iii) $$e^{z+w} = \sum_{n=0}^{\infty} \frac{1}{n!} \boxed{(z+w)^n}$$

$$\uparrow$$

$$(z+w)^n = \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\cancel{n!}} \sum_{k=0}^{n} \frac{\cancel{n!}}{k!\,(n-k)!} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$$

$$\overset{?}{=} \left(\sum_{j=0}^{\infty} \frac{z^j}{j!} \right) \left(\sum_{\ell=0}^{\infty} \frac{w^\ell}{\ell!} \right)$$

$$\sum_{n=0}^{\infty} \sum_{j,\ell:} \frac{z^j}{j!} \frac{w^\ell}{\ell!}$$

$j+l=n$

iv) $$\overline{e^{z}} = \overline{\sum \frac{z^n}{n!}} = \sum \overline{\frac{z^n}{n!}}$$

$$= \sum \overset{\cancel{}}{\frac{1}{n!}} \; \underset{\bar{z}^{n}}{\underset{\|}{\overline{z^n}}} = e^{\bar{z}}.$$

■

Def:

$$\operatorname{Sh} z = \frac{e^{z} - e^{-z}}{2}, \quad \operatorname{Ch} z = \frac{e^{z} + e^{-z}}{2}.$$

$z \in \mathbb{C}$.

Rmk: $$\operatorname{Sh} z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad \operatorname{Ch} z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

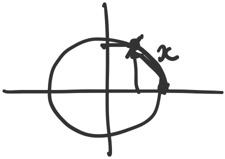
Sin / Cos in $\mathbb{C}$

Recall that

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o\left(x^{2n+1}\right)$$

Thm:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad x \in \mathbb{R}.$$



$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Def:

$$\sin z := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad (\text{sin series})$$

$$\cos z := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \qquad z \in \mathbb{C}. \quad (\text{cos series})$$

Prop: sin / cos series have both $R = +\infty$.

Moreover:

i) $\sin z = -i \,\mathrm{Sh}(iz)$

$\cos z = \mathrm{Ch}(iz)$

ii) (Euler formulas)

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

iii) $\cos z + i \sin z = e^{iz}$

(in part, if $z \in \mathbb{R}$ $z = y$

$$e^{iy} = \cos y + i \sin y.)$$

$\mathbb{C}$ - logs

Recall that we say that

$(u, v \in \mathbb{R})$ $\quad u = \log v \iff e^{u} = v$

Def: Given $w \in \mathbb{C}$ we say that $z \in \mathbb{C}$ is log of w if

$$e^{z} = w.$$

w = known

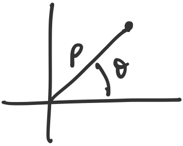
z = unknown $\quad z = x + iy$

$$e^{z} = w \iff e^{x+iy} = w$$

$$\iff e^{x} e^{iy} = w$$

$$\Leftrightarrow \quad \underbrace{e^{x}(\cos y + i \sin y)}_{\text{trig form}} = w$$

$$\underset{\substack{\vee \\ 0}}{\rho}(\cos\theta + i\sin\theta)$$



So if $w = \textcircled{$\rho$}(\cos\theta + i\sin\theta)$ we have that

$$e^{z} = w \quad \Leftrightarrow \quad e^{x}(\cos y + i\sin y) = \rho(\cos\theta + i\sin\theta)$$

$$\Leftrightarrow \begin{cases} e^{x} = \rho \\ y = \theta + k2\pi \qquad k \in \mathbb{Z} \end{cases}$$

$$\overset{\rho>0}{\Leftrightarrow} \begin{cases} x = \log\rho \\ y = \theta + k2\pi \qquad k \in \mathbb{Z}. \end{cases}$$

Prop: If $w = \rho(\cos\theta + i\sin\theta) \neq 0_{\mathbb{C}}$,

$z = x + iy \in \mathbb{C}$ is s.t $e^z = w$

$$\iff \begin{cases} x = \log \rho \\ y = \theta + k2\pi, & k \in \mathbb{Z} \end{cases}$$

We set

$$\log w = \log \rho + i(\theta + k2\pi) \quad k \in \mathbb{Z}$$

We call principal log the funct

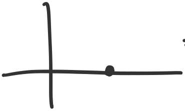
$$\log_{\mathbb{C}} w := \log_{\mathbb{R}}|w| + i \underbrace{\arg(w)}_{\in [0, 2\pi[}$$

Examples:

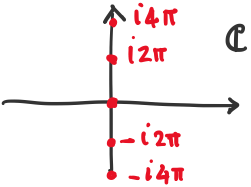
Solve

i) $e^z = 1$

Sol: $z = \log 1 = \overset{\overset{0}{\shortparallel}}{\log_{\mathbb{R}}|1|} + i(\overset{\overset{0}{\shortparallel}}{\arg 1} + k2\pi)$ $\quad k \in \mathbb{Z}$



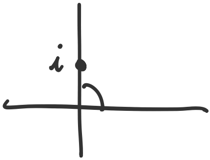
$$= i\underbrace{k2\pi}_{} \quad k \in \mathbb{Z}$$



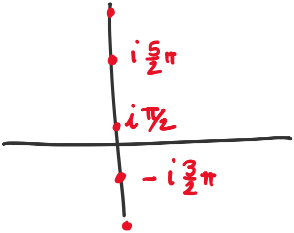
ii) $e^{z} = \underset{\overset{\|}{w}}{i}$

$$z = \log w = \log_{\mathbb{R}} \overset{\overset{\overset{\overset{1}{\|}}{|i|}}{\|}}{|w|} + i(\arg w + k2\pi)$$

$$= \underset{\underset{0}{\|}}{\log 1} + i\left(\frac{\pi}{2} + k2\pi\right)$$



$$= i\left(\frac{\pi}{2} + k2\pi\right) \qquad k \in \mathbb{Z}$$



iii) $\mathrm{Ch}\, z = 0$

$$\mathrm{Ch}\, z = \frac{e^{z} + e^{-z}}{2} = 0$$

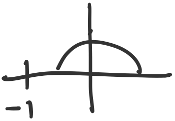
$$\Leftrightarrow \quad e^{z} + e^{-z} = 0$$

$$\Leftrightarrow \quad e^{2z} + 1 = 0$$

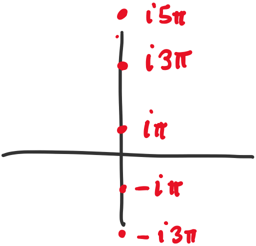
$$\Leftrightarrow \quad e^{2z} = -1$$

$$\Leftrightarrow \quad 2z = \log(-1) = \log_{\mathbb{R}} \overset{1}{\overset{\shortparallel}{|-1|}} + i\left(\arg(-1) + k2\pi\right)$$

$$= 0 + i(\pi + k2\pi)$$



$$= i(2k+1)\pi$$



v) $\sin z = 4$

$\shortparallel$

$$\frac{e^{iz} - e^{-iz}}{2i} = 4 \quad \Leftrightarrow \quad e^{iz} - e^{-iz} = 8i$$

Rmk: $e^{z} \neq 0 \quad \forall z \in \mathbb{C}$

$$\left(e^{z}, e^{-z} \Rightarrow e^{z} \cdot e^{-z} = e^{z-z} = e^{0} = 1 \right.$$

$$\left. \Rightarrow e^{z} \neq 0 \quad \forall z \in \mathbb{C} \right)$$

$$\Leftrightarrow \quad e^{2iz} - 1 = 8i\, e^{iz}$$

$$\Leftrightarrow \quad (e^{iz})^{2} \boxed{-\,8i}\, e^{iz} - 1 = 0$$

$$e^{iz} = \frac{8i \pm \sqrt{-64+4}}{2}$$

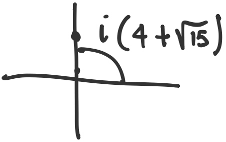
$$= \frac{8i \pm i\sqrt{60}}{2} = i\left(4 \pm \cancel{2}\sqrt{15}\right)$$

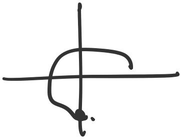
$$e^{iz} = i\left(4+\sqrt{15}\right) \qquad e^{iz} = i\left(4-\sqrt{15}\right)$$

$$\Updownarrow$$

$$iz = \log\left(i\left(4+\sqrt{15}\right)\right)$$

$$= \log_{\mathbb{R}} \left| \;\;'' \;\; \right| + i\left(\arg \qquad + k2\pi\right)$$



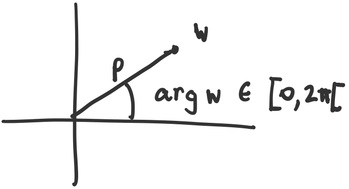


$$iz = \log(4+\sqrt{15}) + i\left(\frac{\pi}{2}+k2\pi\right)$$

$$z = \frac{1}{i}\log(4+\sqrt{15}) + \left(\frac{\pi}{2}+k2\pi\right) \qquad k\in\mathbb{Z}$$

$$+\ \boxed{\begin{array}{l} z = -i\log(4+\sqrt{15}) + \left(\frac{\pi}{2}+k2\pi\right) \quad k\in\mathbb{Z} \\ \\ z = -i\log(4-\sqrt{15}) + \left(\frac{\pi}{2}+k2\pi\right) \quad k\in\mathbb{Z}. \end{array}}$$

Do 6.10.2



ℂ – Differentiability

For functs $f=f(z) : D\subset\mathbb{C}\longrightarrow\mathbb{C}$

we can def a derivative in the

same way we do for functs of real variable:

Def: We say that f is $\mathbb{C}$-diff. at some pt z if

$$\exists \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} =: f'(z) \in \mathbb{C}.$$

Examples:

$$f \equiv c \implies f' \equiv 0$$

$$f'(z) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \overset{\quad 0}{\frac{0}{h}} = 0.$$

$$f(z) = z^n \implies f'(z) = n z^{n-1}$$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{(z+h)^n - z^n}{h}$$

$$(z+h)^n = \underset{\overset{\shortparallel}{1}}{\binom{n}{0}} z^{n} h^{0} + \underset{\overset{\shortparallel}{n}}{\binom{n}{1}} z^{n-1} h^{1} + \binom{n}{2} z^{n-2} h^{2} + \dots$$

$$+ \dots + \binom{n}{n} z^{0} h^{n}$$

$$= z^n + n z^{n-1} h + h^2 \left(\_ \;\; \_\_ \;\; \_ \right)$$

$$\frac{(z+h)^n - z^n}{h} = n z^{n-1} + h \left(\_\_ \;\; \_ \, . \right)$$

$$\xrightarrow[h \to 0]{} n z^{n-1}$$

$$f(z) = \frac{1}{z^n} \qquad z \neq 0$$

$$= z^{-n}$$

$$f'(z) = -n z^{-n-1}$$

$$= -\frac{n}{z^{n+1}} \, .$$

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