5.7.5 #5

$$I := \int_{\underline{\mathbb{R}^3}} \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz$$

Sol:

$$I \underset{\text{sph. coords}}{:=} \int_{\substack{\rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi}} \boxed{\rho\, e^{-\rho^2} \cdot \rho^2 \sin\varphi}\; d\rho\, d\theta\, d\varphi$$

$$\overset{RF}{=} \int_{\substack{\rho \geq 0 \\ 0 \leq \varphi \leq \pi}} \underbrace{\rho^3 e^{-\rho^2}} \sin\varphi \underbrace{\int_0^{2\pi} 1 \; d\theta}_{\substack{\parallel \\ 2\pi}} \; d\rho\, d\varphi$$

$$\overset{RF}{=} 2\pi \int_0^{+\infty} \rho^3 e^{-\rho^2} \underbrace{\int_0^{\pi} \sin\varphi \; d\varphi}_{\substack{\parallel \\ -\cos\varphi \big|_0^{\pi} = -(-1-1) = +2}} \; d\rho$$

$$= 2\cancel{4}\pi \int_0^{+\infty} 2\rho^3 e^{-\rho^2}\, d\rho$$

$$\rho^2 \cdot \underbrace{(-2\rho\, e^{-\rho^2})}_{\partial_\rho e^{-\rho^2} = e^{-\rho^2} \cdot (-2\rho)}$$

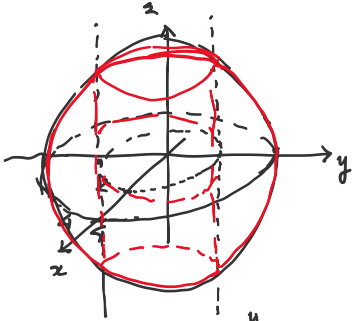
$$= -2\pi \left[\underbrace{\left[\rho^2 e^{-\rho^2} \right]_{\rho=0}^{\rho=+\infty}}_{0-0=0} + \int_0^{+\infty} -2\rho e^{-\rho^2} d\rho \right]$$

$$= -2\pi \underbrace{\left[+ e^{-\rho^2} \right]_{\rho=0}^{\rho=+\infty}}_{0-1} = 2\pi.$$

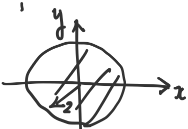
■

5.7.4 Compute the vol of

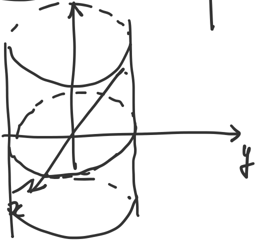
$$\# \left\{ (x,y,z) \in \mathbb{R}^3 : \underbrace{x^2+y^2+z^2 \leq 16,}\ \overbrace{x^2+y^2 \geq 4} \right\}$$



$$\boxed{x^2+y^2 \leq 4}$$



z



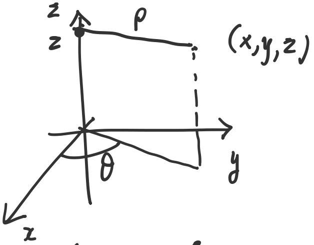
$$\mathrm{Vol}\, D = \lambda_3(D) = \int_D 1 \quad dx\,dy\,dz$$

$$= \int_{\underbrace{\substack{x^2+y^2+z^2 \leq 16 \\ x^2+y^2 \geqslant 4}}} 1 \quad dx\,dy\,dz$$

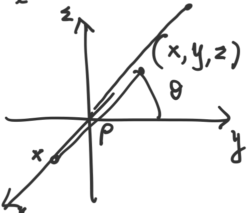
$$\underset{\text{cyl coords}}{=} \int 1 \cdot \rho \quad d\rho\, d\theta \quad dz$$

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = z \end{cases}$$

$$x^2+y^2 = \rho^2$$



$$\begin{cases} x = x \\ y = \rho \cos\theta \\ z = \rho \sin\theta \end{cases}$$



$y^2+z^2=\rho^2$

$$\underset{\text{cyl coords}}{=} \int_{\substack{\rho^2 \geq 4 \iff \rho \geq 2 \\ \rho^2+z^2 \leq 16 \\ 0 \leq \theta \leq 2\pi}} \boxed{1 \cdot \rho} \, d\rho \, d\theta \, dz$$

$$\overset{RF}{=} \int_{\substack{\rho \geq 2 \\ \rho^2+z^2 \leq 16}} \rho \int_0^{2\pi} 1 \, d\theta \, d\rho \, dz$$

$$= 2\pi \int_{\substack{\rho \geq 2 \\ \rho^2+z^2 \leq 16}} \rho \, d\rho \, dz$$

$$= 2\pi \int_2^4 \rho \int_{-\sqrt{16-\rho^2}}^{\sqrt{16-\rho^2}} \cancel{\rho}\, 1 \, dz \, d\rho$$

$$\boxed{\rho \geq 2}$$
$$\rho^2+z^2 \leq 16 \leftarrow \qquad z^2 \leq 16-\rho^2$$

$$\iff \boxed{-\sqrt{16-\rho^2} \leq z \leq \sqrt{16-\rho^2}}$$

$$16-\rho^2 \geq 0 \iff \rho^2 \leq 16$$
$$\Updownarrow$$
$$\boxed{\rho \geq 2} \qquad \boxed{0 \leq \rho \leq 4}$$

$$2 \leqslant \rho \leqslant 4$$

$$= \frac{4\pi}{-3} \int_2^4 \overset{-3}{\cancel{2}}\rho \sqrt{16-\rho^2} \, d\rho$$

$$2\rho(16-\rho^2)^{1/2}$$

$$\partial_\rho (16-\rho^2)^{3/2} = \frac{3}{\cancel{2}}(16-\rho^2)^{1/2} \cdot (-\cancel{2}\rho)$$

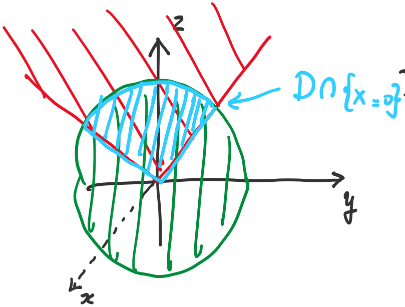
$$= \ominus \, \frac{4}{3}\pi \left[(16-\rho^2)^{3/2} \right]_{\rho=2}^{\rho=4}$$

$$0 \ominus 12^{3/2}$$

$$= \frac{4}{3} 12^{3/2} \pi. \quad \blacksquare$$

#7.

$$D = \{ (x,y,z) \in \mathbb{R}^3 : z \geqslant \sqrt{x^2+y^2}, \; x^2+y^2+z^2 \leqslant 1 \}$$

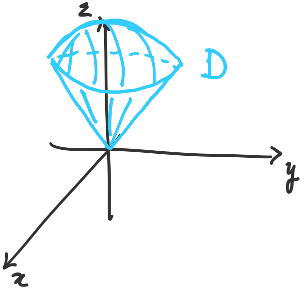


$$D \cap \text{plane } yz \; \| \; \{x=0\}$$

$$D \cap \{x=0\} = \{ (0,y,z) \in \mathbb{R}^3 : z \geqslant \sqrt{y^2} = |y|$$

$y^2+z^2 \leq 1 \}$

$= \{ z \geq |y|, \quad y^2+z^2 \leq 1 \}$



$$\text{Vol } D = \int_D 1 \; dx\,dy\,dz = \int_{\substack{z \geq \rho, \\ \rho^2+z^2 \leq 1 \\ 0 \leq \theta \leq 2\pi}} 1 \cdot \rho \; d\rho\, d\theta\, dz$$

$=$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

$$\overset{RF}{=} \int \quad \overset{\overset{2\pi}{\shortparallel}}{\int_0^{2\pi} d\theta} \quad d\rho\, dz$$

$$= 2\pi \int \underline{\quad \rho \;} \, d\rho\, dz$$

$$\boxed{\begin{array}{l} z \geqslant \rho \\ \rho^2 + z^2 \leqslant 1 \end{array}}$$

$$\text{RF} = 2\pi \int_0^{\cancel{1}\ 1/\sqrt{2}} \rho \int_\rho^{\sqrt{1-\rho^2}} 1 \, dz \, d\rho$$

$$\begin{array}{l} z \geqslant \rho \\ \rho^2 + z^2 \leqslant 1 \end{array} \Rightarrow \begin{array}{l} \boxed{z \geqslant \rho} \geqslant 0 \\ z^2 \leqslant 1-\rho^2 \Leftrightarrow -\sqrt{1-\rho^2} \leqslant z \leqslant \sqrt{1-\rho^2} \\ \qquad - \qquad \rho^2 \leqslant 1 \end{array}$$

$$<$$

$$\Leftrightarrow \quad \boxed{\rho \leqslant z \leqslant \sqrt{1-\rho^2}} \quad 0 \leqslant \rho \leqslant 1$$

$$\Leftrightarrow \quad \begin{array}{l} \rho \leqslant z \leqslant \sqrt{1-\rho^2} \\ 0 \leqslant \rho \leqslant 1 \\ \rho \leqslant \sqrt{1-\rho^2} \\ \quad \Updownarrow + \\ \rho^2 \leqslant 1-\rho^2 \Leftrightarrow 2\rho^2 \leqslant 1 \end{array}$$

$$\Leftrightarrow \quad \rho^2 \leqslant 1/2$$

$$\Leftrightarrow \quad 0 \leqslant \rho \leqslant \frac{1}{\sqrt{2}}$$

$$= 2\pi \int_0^{1/\sqrt{2}} \rho \int_\rho^{\sqrt{1-\rho^2}} 1 \, dz \, d\rho$$

$$\|$$

$$\left(\sqrt{1-\rho^2} - \rho\right)$$

$$= 2\pi \int_0^{1/\sqrt{2}} \rho \left(\sqrt{1-\rho^2} - \rho\right) d\rho$$

$$= 2\pi \int_0^{1/\sqrt{2}} \rho (1-\rho^2)^{1/2} - \boxed{\rho^2}\, d\rho$$

$$\partial_\rho (1-\rho^2)^{3/2} = \frac{3}{\cancel{2}} (1-\rho^2)^{1/2} \cdot (-\cancel{2}\rho)$$

$$= 2\pi \left[\ominus \frac{1}{3} \int_0^{1/\sqrt{2}} -3\rho(1-\rho^2)^{1/2}\, d\rho \ominus \frac{\rho^3}{3}\Big|_0^{1/\sqrt{2}} \right]$$

$$\parallel$$

$$\left[(1-\rho^2)^{3/2} \right]_0^{1/\sqrt{2}}$$

$$= -\frac{1}{3} 2\pi \left[\frac{1}{2^{3/2}} - 1 + \frac{1}{2^{3/2}} - 0 \right]$$

$$= \frac{2}{3}\pi \left[1 - \frac{2}{2^{3/2}} \right] = \frac{2}{3}\pi \left(1 - \frac{1}{\sqrt{2}} \right). \quad ■$$

Do # 6.5

#1 $D = \left\{ (x,y,z) : 9\left(1-\sqrt{x^2+y^2}\right)^2 + 4z^2 \leq 1 \right\}$

$$\text{Vol } D = \int_D 1 \, dx\,dy\,dz$$

(try to do the fig of D by drawing first $D \cap \{x=0\}$, then rot. around z-axis)

$$\underset{\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}}{=} \int_{\substack{9(1-\rho)^2+4z^2 \leq 1 \\ 0 \leq \theta \leq 2\pi}} 1 \cdot \rho \, d\rho d\theta dz$$

$$\overset{RF}{=} 2\pi \int_{9(1-\rho)^2+4z^2 \leq 1} \rho \, d\rho \, dz$$

$$\overset{(*)}{=} 2\pi \int \int$$

$$\boxed{9(1-\rho)^2+4z^2 \leq 1} \iff \cancel{4}z^2 \leq \frac{1-9(1-\rho)^2}{4}$$

$$\iff$$

$$-\sqrt{\frac{1-9(1-\rho)^2}{4}} \leq z \leq \sqrt{\frac{1-9(1-\rho)^2}{4}}$$

$$1-9(1-\rho)^2 \geq 0$$

$$\iff \cancel{9}(1-\rho)^2 \leq 1/9$$

$$-1/3 \leq 1-\rho \leq 1/3$$

$$2/3 = 1-1/3 \leq \rho \leq 1+1/3 = 4/3$$

$$\Rightarrow \overset{(*)}{=} 2\pi \int_{2/3}^{4/3} \rho \int_{-\sqrt{\frac{1-9(1-\rho)^2}{4}}}^{\sqrt{\frac{1-9(1-\rho)^2}{4}}} 1 \quad dz \quad d\rho \overset{(**)}{=}$$

What if I look for a cond on ρ first and z second

$$\overset{(*)}{=} 2\pi \int_{-1/2}^{1/2} \int_{1-\sqrt{\quad}}^{1+\sqrt{\quad}} \rho \; d\rho \; dz$$

$$\cancel{9}(1-\rho)^2 \leqslant \frac{1-4z^2}{9}$$

$$-\sqrt{\frac{1-4z^2}{9}} \leqslant 1-\rho \leqslant \sqrt{\frac{1-4z^2}{9}}$$

$$1-\sqrt{\frac{1-4z^2}{9}} \leqslant \rho \leqslant 1+\sqrt{\frac{1-4z^2}{9}}$$

Provided $1-4z^2 \geqslant 0 \iff z^2 \leqslant 1/4$

$\iff -1/2 \leqslant z \leqslant 1/2$

$$1-\sqrt{\frac{1-4z^2}{9}} < 0 \iff 1 < \sqrt{\quad}$$

$$\iff \quad 1 < \frac{1-4z^2}{9} \quad \text{never}$$

$$\overset{(**)}{=} \quad 2\pi \int_{2/3}^{4/3} \rho \, \cancel{2} \frac{1}{\cancel{2}} \sqrt{1-9(1-\rho)^2} \; d\rho$$

$$= -2\pi \int_{2/3}^{4/3} -1+1-\rho \left(1-9(1-\rho)^2\right)^{1/2} \; d\rho$$

$$\partial_\rho \left(1-9(1-\rho)^2\right)^{3/2} = \frac{27\,\cancel{3}}{\cancel{2}} \left(1-9(1-\rho)^2\right)^{1/2} \left(+9 \cdot \cancel{2}(1-\rho)\cdot(+1)\right)$$

$$= +2\pi \int_{2/3}^{4/3} \sqrt{1-\underbrace{9(1-\rho)^2}_{(3(1-\rho))^2}} \; d\rho + 2\pi \int_{2/3}^{4/3} (1-\rho)\left(1-9(1-\rho)^2\right)^{1/2}$$

$$3(1-\rho) = \sin\alpha$$

long, you finish.....

$$\int \sqrt{1-u^2} \; du \; ...$$

Do 5.7.4 # 2 (use cyl coord, draw D)
(a bit difficult....)

5.7.6 # 2,3

5.7.7

5.7.8
5.7.9

Holomorphic Functions

We consider now functs

$$f = f(z) \; : \; D \subset \mathbb{C} \longrightarrow \mathbb{C}$$

Examples:

$$f \equiv c \qquad c \in \mathbb{C} \qquad (\text{const.})$$

$$f(z) = z$$

$$f(z) = z^2$$

$$f(z) = z^n$$

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n$$

(polynomial)

$$f(z) = \frac{p(z)}{q(z)} \qquad (\text{rational funct}).$$

$$f(z) = \bar{z}$$

$$f(x+iy) = x - iy \qquad x, y \in \mathbb{R}$$

In general, $f(z)$ is complex $\Rightarrow$

$$f(z) = \underbrace{u(z)}_{\mathrm{Re}\, f(z)} + i\, \underbrace{v(z)}_{\mathrm{Im}\, f(z)}$$

$z = x + iy$

$$\boxed{f(x+iy) = u(x,y) + i\, v(x,y)}$$

$$u(x,y) := \mathrm{Re}\, f(x+iy)$$

$$v(x,y) := \mathrm{Im}\, f(x+iy)$$

Ex:

$$f(x+iy) = x^2 y + i\, x y^2$$

$$f(x+iy) = (\sin x) e^{y} + i \cos(\log(x+y))$$

An important class of functs $\mathbb{C} \to \mathbb{C}$ is the class of power series:

$$f(z) := \sum^{\infty} c_n z^n$$

$n=0$

$$(c_n) \subset \mathbb{C}$$

Ex: $f(z) = \sum_{n=0}^{\infty} z^n$ (geometric series)

Recall that

$$S_N = \sum_{n=0}^{N} z^n = \overset{S_{N-1}}{\boxed{1 + z + z^2 + z^3 + \cdots z^{N-1}}} + z^N$$

$$= 1 + z\,(\underbrace{1 + z + z^2 + \cdots + z^{N-1}}_{S_{N-1}})$$

$$\Rightarrow \quad S_N = 1 + z\,S_{N-1}$$

$$S_N = S_{N-1} + z^N \quad \Rightarrow \quad S_{N-1} = S_N - z^N$$

$$S_N = 1 + z\,(S_N - z^N)$$

$$\Rightarrow \quad (1-z)\,S_N = 1 - z^N$$

$$\Rightarrow \quad S_N = \sum_{n=0}^{N} z^n = \frac{1-z^N}{1-z}$$

$$N \to +\infty \qquad \sum_{n=0}^{\infty} z^n = \lim_{N \to \infty} \sum_{n=0}^{N} z^n$$

$$= \lim_{N \to \infty} \frac{1-z^N}{1-z}$$

$$z^N \to \begin{cases} \boxed{0 \qquad |z| < 1} \\ \nexists \qquad |z| \geq 1 \end{cases}$$

$$= \frac{1}{1-z}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad |z| < 1$$

This means that : $f(z) = \sum_{n=0}^{\infty} z^n$ is well defd on $D \subset \mathbb{C}$, $D = \{z \in \mathbb{C} : |z| < 1\}$ and

$$f(z) = \frac{1}{1-z} . \qquad \blacksquare$$

In gen, a power series is def. when the series converges. Applying ratio test

⌈ $\sum_{n=0}^{\infty} a_n$, $a_n > 0$ $\qquad q = \lim \frac{a_{n+1}}{a_n} \in [0, +\infty]$

if $q < 1 \Rightarrow \sum a_n$ conv

$q > 1 \Rightarrow \sum a_n$ div and $a_n \to +\infty$

$q = 1$?

⌋

to $\sum \underbrace{|c_n z^n|}$

a_n

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|c_{n+1} z^{\cancel{n}+1}|}{|c_n \cancel{z^n}|}$$

$$q = |z| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

So

if $q = |z| \lim \left| \frac{c_{n+1}}{c_n} \right| < 1$

$\Rightarrow \quad \sum |c_n z^n| \quad$ conv

$\Rightarrow \quad \sum c_n z^n \quad$ conv

if $q = |z| \lim \left| \frac{c_{n+1}}{c_n} \right| > 1$

$\Rightarrow \quad \sum |c_n z^n| \quad$ diverges

$\Big(\sum (-1)^n \frac{1}{n} \quad$ conv

$\sum | \quad | = \sum \frac{1}{n} \quad$ div $\Big)$

$+ \quad |c_n z^n| \longrightarrow +\infty$

$\quad \not\downarrow \qquad$ nec. cond for convergence

$\quad 0 \qquad$ fails

$\Rightarrow \quad \sum c_n z^n \quad$ does not conv.

So, the $\sum c_n z^n$ conv if

$$q = |z| \lim \left| \frac{c_{n+1}}{c_n} \right| < 1$$

$$\Leftrightarrow \quad |z| < \frac{1}{\lim \left| \frac{c_{n+1}}{c_n} \right|}$$

$$\Leftrightarrow \quad \boxed{|z| < \lim \left| \frac{c_n}{c_{n+1}} \right|}$$

the $\sum c_n z^n$ does not conv if

$$|z| > \lim \left| \frac{c_n}{c_{n+1}} \right|$$

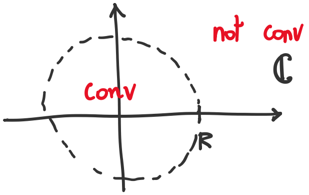
Prop: Let $\sum c_n z^n$ be a power series

$$R := \lim_{n \to +\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

We have $\sum c_n z^n$

converges for z : $|z| < R$

does not conv. " " : $|z| > R$



If $R = +\infty \Rightarrow \sum c_n z^n$ conv in the entire $\mathbb{C}$.

We call R radius of conv.

Do 6.10.1