COMPUTABILITY (08/01/2024)

EXERCISE: Given $f: \mathbb{N} \to \mathbb{N}$ a fixed function and define

$$B_f = \{ e \in \mathbb{N} \mid \varphi_e = f \}$$

The set B_f is saturated

$$B_f = \{ e \in \mathbb{N} \mid \varphi_e \in \mathcal{B}_f \} \qquad \mathcal{B}_f = \{ f \}$$

2 cases

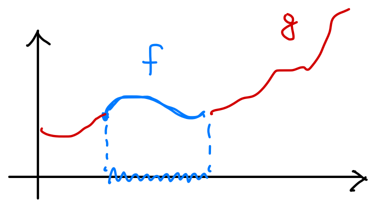
1. f is not computable

$B_f = \emptyset \qquad \overline{B_f} = \mathbb{N}$ recursive

2. f is computable

* B_f is not r.e.

(2.a) if f is finite $\qquad f = \vartheta$



let g be a total function s.t. $\quad f \subseteq g$, $f \neq g$

then $g \notin \mathcal{B}_f$ and $\underset{\text{finite}}{f = \vartheta} \subseteq g$ and $f \in \mathcal{B}_f$

hence by Rice - Shapiro B_f not r.e. (hence not recursive)

(2.b) f is not finite

note that $\quad f \in \mathcal{B}_f \quad$ and $\quad \forall \vartheta \subseteq f \quad \vartheta$ finite $\vartheta \notin \mathcal{B}_f$

then B_f not r.e. by Rice - Shapiro (hence B_f not recursive)

* $\overline{B_f}$

* $f = \emptyset \qquad (f(x) \uparrow \quad \forall x)$

$\overline{B_f}$ is r.e. since

$e \in \overline{B_f}$ iff there is some input $\varphi_e(x)\downarrow$

hence

$$SC_{\overline{B_f}}(e) = \mathbb{1}\left(\mu(x,t).\ H(e,x,t)\right)$$

$$= \mathbb{1}\left(\mu\omega.\ H(e,(\omega)_1,(\omega)_2)\right)$$

computable hence $\overline{B_f}$ r.e. (not recursive since B_f not r.e. hence not recursive)

* $f \neq \emptyset$

$\overline{B_f}$ is not r.e. by Rice-Shapiro

$f \notin \overline{B_f}$ and $\vartheta = \emptyset \subseteq f$ $\quad \vartheta \notin B_f$ hence $\vartheta \in \overline{B_f}$

hence by Rice-Shapiro $\overline{B_f}$ is not r.e.

$\square$

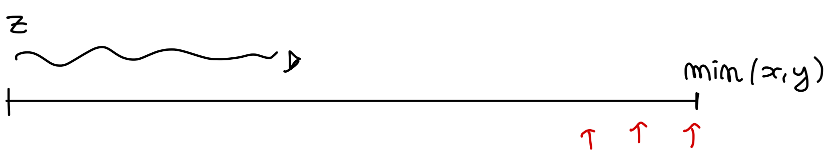
<u>EXERCISE</u>: show that $gcd : \mathbb{N}^2 \to \mathbb{N}$

$gcd(x,y)$ = greatest common divisor of x and y

is computable (primitive recursive)

$$gcd(x,y) = \max z.\ \underset{\displaystyle rm(x,z)=0}{\underset{|||}{z \text{ divisor of } x}} \text{ and } \underset{\displaystyle rm(z,y)=0}{\underset{|||}{z \text{ divisor of } y}}$$

$$= \max z \leq \min(x,y).\ \left(rm(z,x) + rm(z,y) = 0\right)$$



$$z = \min(x,y) - \omega$$

$\uparrow$ largest $\qquad$ $\uparrow$ least

$$= \min(x,y) - \left(\mu\omega \leq \min(x,y).\ \left(z = \min(x,y) - \omega \ \wedge\ rm(z,x) + rm(z,y) = 0\right)\right)$$

$$= \underbrace{\min(x,y) \dot{-} \underbrace{\left(\mu w \leq \min(x,y) . \underbrace{\left(rm(\min(x,y) \dot{-} w, x) + rm(\min(x,y) \dot{-} w, y)\right)}_{PR}\right)}_{PR}}_{PR}$$

hence gcd is primitive recursive.

<u>EXERCISE</u> : show that there are $m, n \in \mathbb{N}$

(i) $\varphi_n = \varphi_{n+1}$

(ii) $\varphi_m \neq \varphi_{m+1}$

(i) observe that $s(n) = n+1$ is total and computable, hence by the second recursion theorem there is $n \in \mathbb{N}$

$$\varphi_n = \varphi_{s(n)} = \varphi_{n+1}$$

(ii) if it were that $\forall m$ $\varphi_m = \varphi_{m+1}$

then inductively $\varphi_0 = \varphi_1 = \varphi_2 = \varphi_3 = \dots$

i.e. all computable unary functions would be the same

and this is not the case (e.g. $\mathbb{1} \neq succ$)

EXERCISE : Define PR. Using only the definition show that

$$max_2 : \mathbb{N} \to \mathbb{N} \quad \text{is PR}$$
$$max_2(x) = max(2, x)$$

Two ways:

① Rebuild max

sum $\quad x+y$

$$\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases}$$

predecessor $\quad y \dot{-} 1$

$$\begin{cases} 0 \dot{-} 1 = 0 \\ y+1 \dot{-} 1 = y \end{cases}$$

difference $\quad x \dot{-} y$

$$\begin{cases} x \dot{-} 0 = x \\ x \dot{-} (y+1) = (x \dot{-} y) \dot{-} 1 \end{cases}$$

max $\quad max(x,y) = x + (y \dot{-} x)$

$$max_2(x) = max(2, x) = max(((0+1)+1), x)$$

② Define what you really need

$$max_2(0) = 2$$

$$max_2(y+1) = \begin{cases} 2 = 1+1 & \text{if } y=0 \\ y+1 & \text{if } y>0 \end{cases} = y+1+\overline{sg}(y)$$

max (1, y)

sum as above

$sg(y)$

$$\overline{sg}(0) = 1$$

$$\overline{sg}(y+1) = 0$$

③ Even faster ...

$$\begin{cases} max_2(0) = 2 \\ max_2(y+1) = max_1(y) + 1 \end{cases}$$

$$max_1(y) = max(1, y)$$

$$\begin{cases} max_1(0) = 1 \\ max_1(y+1) = y+1 \end{cases}$$

EXERCISE :

① $A = \{ x \mid \varphi_x(x) = x^2 \}$

② $B = \{ x \mid \varphi_x(y) = y^2$ for infinitely many y's $\}$

Questions :

(a) Classify A, B according to recursiveness

(b) Are A, B saturated ?

① $A = \{ x \mid \varphi_x(x) = x^2 \}$

conjecture $\left. \begin{array}{l} A \text{ r.e.} \\ \text{not recursive} \end{array} \right\} \rightarrow \bar{A}$ not r.e. (hence not recursive)

* A r.e.

$$SC_A(x) = 1\!\!1 \big(\underbrace{\mu z. \mid \varphi_x(x) - x^2 \mid}_{\begin{array}{l} 0 \text{ if } \varphi_x(x) = x^2 \\ \uparrow \text{ otherwise} \end{array}} \big) = 1\!\!1 \left(\mu z. \mid \Psi_U(x,x) - x^2 \mid \right)$$

$\hookrightarrow 0$ if $\varphi_x(x) = x^2$

$\hookrightarrow \neq 0$ otherwise

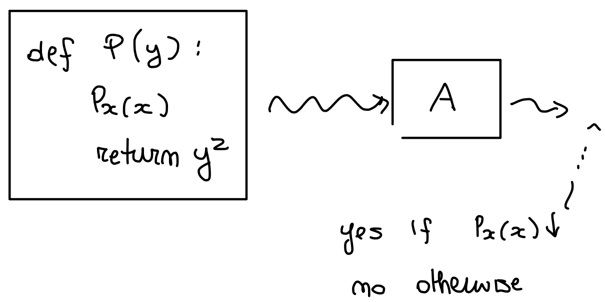
$\uparrow$

computable

$\leadsto$ A r.e.

* A is not recursive

given x



define

$$g(x,y) = \mathbb{1}(\varphi_x(x)) \cdot y^2$$
$$= \mathbb{1}(\varphi_U(x,x))\, y^2 = \begin{cases} y^2 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

computable, hence by smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ total and computable s.t.

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} y^2 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

and s is the reduction function for $K \leq_m A$

* if $x \in K$ then $\varphi_{s(x)}(y) = y^2 \ \forall y$, hence in particular
$$\varphi_{s(x)}(s(x)) = (s(x))^2$$

$\rightsquigarrow$ $s(x) \in A$

* if $x \notin K$ then $\varphi_{s(x)}(y)\uparrow \ \forall y$ hence $\varphi_{s(x)}(s(x)) \neq (s(x))^2$

$\rightsquigarrow$ $s(x) \notin A$

hence $K \leq_m A$ and since K not recursive, A is not recursive

(b) $A = \{ x \mid \varphi_x(x) = x^2 \}$ saturated? NO

let $e \in \mathbb{N}$ s.t. $\varphi_e(x) = \begin{cases} e^2 & \text{if } x = e \\ \uparrow & \text{otherwise} \end{cases}$

note that such e exists.

In fact define

$$g(z,x) = \begin{cases} z^2 & x = z \\ \uparrow & \text{otherwise} \end{cases}$$

$$= z^2 + \underbrace{\mu w.\ |x - z|}_{\begin{cases} 0 & \text{if } x = z \\ \uparrow & \text{otherwise} \end{cases}}$$

computable

By smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ total computable s.t

$$\varphi_{s(z)}(x) = g(z,x) = \begin{cases} z^2 & \text{if } x = z \\ \uparrow & \text{otherwise} \end{cases}$$

By the 2nd recursion theorem there is $e \in \mathbb{N}$ s.t. $\varphi_e = \varphi_{s(e)}$

$$\varphi_e(x) = \varphi_{s(e)}(x) = g(e,x) = \begin{cases} e^2 & \text{if } x = e \\ \uparrow & \text{otherwise} \end{cases}$$

Given this

* $\boxed{e \in A}$ (since $\varphi_e(e) = e^2$)

* let $e' \neq e$ s.t $\boxed{\varphi_e = \varphi_{e'}}$

$$\varphi_{e'}(e') = \varphi_e(e') \uparrow \neq e'^2$$

$\Rightarrow$ $\boxed{e' \notin A}$

$\leadsto$ A not saturated

(2) $B = \{ x \mid \varphi_x(y) = y^2 \text{ for infinitely many } y\text{'s} \}$

B is saturated

$B = \{ x \mid \varphi_x \in \mathcal{B} \}$

$\mathcal{B} = \{ f \mid f(y) = y^2 \text{ for infinitely many } y\text{'s} \}$

conjecture : $B, \overline{B}$ not r.e. (hence not recursive)

* <u>B is not r.e.</u>

let $f : \mathbb{N} \to \mathbb{N}$

$f(y) = y^2$

then $f \in \mathcal{B}$ (since $\{ y \mid f(y) = y^2 \} = \mathbb{N}$ infinite)

for all $\vartheta \subseteq f$ ϑ finite $\{ y \mid \vartheta(y) = y^2 \} = dom(\vartheta)$ finite

$\rightsquigarrow \vartheta \notin \mathcal{B}$

hence by Rice - Shapiro B not r.e.

* <u>$\overline{B}$ not r.e.</u>

note that f as defined above $f \notin \overline{\mathcal{B}}$ and $\vartheta = \emptyset \subseteq f$

$\vartheta \in \overline{\mathcal{B}}$

hence, by Rice - Shapiro $\overline{B}$ not r.e.

$\square$