Monday, 8 January 2024 08:34

Ex 5.7.3

5.7.5 # 1, 2, 3

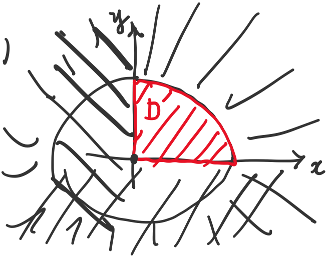
5.7.3 $D = \{(x,y)\in\mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ x^2+y^2 \leqslant r^2\}$

i) Draw D and describe D in pol. coords

ii) Det its barycentre and compute

$$\int_D \frac{x+y}{x^2+y^2}\,dx\,dy.$$

Sol: i)



$$D_{pol} = \{(\underset{\substack{\vee\!|\\ 0}}{\rho},\underset{\substack{\cap\\ [0,2\pi]}}{\theta}) : 0 \leqslant \rho \leqslant r,\ 0 \leqslant \theta \leqslant \pi/2\}$$

ii)

Barycentre of $D \subset \mathbb{R}^2$: $(\bar{x},\bar{y})$

where

$$\bar{x} := \frac{1}{\lambda_2(D)} \int_D x \quad dx\,dy$$

$$\vec{y} := \frac{1}{\lambda_2(D)} \int_D y \quad dx\,dy$$

$\bar{x}$: $\lambda_2(D) = \frac{1}{4}\pi r^2$

$$\int_D \textcircled{x} \; dx\,dy =$$

$D \in \mathcal{E}(D)$, D closed & bdd

$$\Rightarrow \exists \underline{\int_D x \, dx\,dy}$$

$$\overset{RF}{=} \int_0^r \left(\int_0^{\sqrt{r^2-y^2}} x \; d\textcircled{x} \right) dy$$

$x \geqslant 0$, $\boxed{y \geqslant 0}$ $x^2 + y^2 \leqslant r^2$

$$\Updownarrow$$

$$0 \leqslant x^2 \leqslant r^2 - y^2$$

$$\Updownarrow$$

$$-\sqrt{r^2-y^2} \leqslant x \leqslant \sqrt{r^2-y^2}$$

$0 \leqslant x \leqslant \sqrt{r^2-y^2}$,

$$\underbrace{0 \leq y \quad , \quad r^2 - y^2 \geq 0 \Leftrightarrow -r \leq y \leq r}_{0 \leq y \leq r}$$

$$= \int_0^r \left[\frac{x^2}{2} \right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \frac{1}{2} \int_0^r (r^2 - y^2) - 0 \quad dy$$

$$= \frac{1}{2} \left(r^3 - \left[\frac{y^3}{3} \right]_{y=0}^{y=r} \right)$$

$$= \frac{1}{2} \left(r^3 - \frac{r^3}{3} \right) = \frac{r^3}{2} \frac{2}{3} = \frac{r^3}{3}$$

$$\Rightarrow \quad \bar{x} = \frac{1}{\frac{1}{4}\pi r^2} \cdot \frac{r^3}{3} = \frac{4}{3} \frac{r}{\pi} .$$

⌐ Alt :

$$\int_D x \, dx\, dy = \int (\rho \cos\theta) \rho \, d\rho \, d\theta$$

$x = \rho \cos\theta \quad 0 \leq \rho \leq r$

$y = \rho \sin\theta \quad 0 \leq \theta \leq \pi/2$

$$= \int_{\substack{0 \leq \rho \leq r \\ 0 \leq \theta \leq \pi/2}} \rho^2 \cos\theta \, d\rho \, d\theta$$

$$\overset{RF}{=} \int_0^r \rho^2 \left(\int_0^{\pi/2} \cancel{\rho^2} \cos\theta \, d\theta \right) d\rho$$

$$+\sin\theta \Big|_0^{\pi/2} = 1 - 0 = 1$$

$$= 1 \int_0^r \rho^2 \, d\rho$$

$$= \frac{\rho^3}{3} \Big|_0^r = \frac{r^3}{3} \quad \lrcorner$$

$$\bar{y} = \frac{1}{\lambda_2(D)} \int_D y \, dx \, dy$$

$$D = \{x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq r^2\}$$

$$\int_D y \, dx \, dy = \int_{\{y \geq 0,\ x \geq 0,\ y^2 + x^2 \leq r^2\}} x \, dy \, dx = \frac{r^3}{3}$$

$$\Rightarrow \bar{y} = \bar{x}.$$

$$\int_D \underset{\underset{f(x,y)}{\overset{!!}{\parallel}}}{\boxed{\frac{x+y}{x^2+y^2}}} \, dx \, dy =$$

Since $f \notin \mathcal{C}(D)$ we cannot say immediately that f is int on D. So, we must verify $\exists \boxed{\int_D |f| \, dx\, dy}$. Since $f \geq 0$ on D,

$$\int_D |f|\,dx\,dy = \int_D f \, dx\, dy \overset{CV}{=} \int_{\substack{0 \le \rho \le r \\ 0 \le \theta \le \pi/2}} \frac{\cancel{\rho}\cos\theta + \cancel{\rho}\sin\theta}{\cancel{\rho^2}} \cancel{\rho} \, d\rho\, d\theta$$

$$\boxed{\exists} \iff \boxed{\exists}$$

$$= \int_{\substack{0 \le \rho \le r \\ 0 \le \theta \le \pi/2}} (\cos\theta + \sin\theta) \, d\rho \, d\theta$$

$$\overset{RF}{=} \int_0^{\pi/2} \int_0^r (\cos\theta + \sin\theta) \, d\rho \, d\theta$$

$$= r \int_0^{\pi/2} \cos\theta + \sin\theta \, d\theta$$

$$= r \left[\sin\theta \Big|_0^{\pi/2} + -\cos\theta \Big|_0^{\pi/2} \right]$$

$$1 - 0 \qquad - (0 - 1)$$

$$1 + 1$$

$$= 2r. \qquad \blacksquare$$

5.7.5.

1. $$\int_{x^2+y^2\leq 4} \sqrt{4-x^2-y^2}\; dx\,dy$$

Sol: $f(x,y) := \sqrt{4-(x^2+y^2)} \in \mathscr{C}(D)$,

$D = \{x^2+y^2 \leq 4\} = B(\vec{0},2]$.

$\Rightarrow \quad \exists \int_D f$

$$\int_D f \underset{\text{pol. coords}}{=} \int_{\substack{\rho^2\leq 4\\ 0\leq\theta\leq 2\pi}} \sqrt{4-\rho^2}\cdot \rho \; d\rho\, d\theta$$

$$= \int_{\substack{0\leq\rho\leq 2\\ 0\leq\vartheta\leq 2\pi}} \rho\,(4-\rho^2)^{1/2}\; d\rho\, d\theta$$

$$\overset{\text{RF}}{=} -\frac{1}{3}\int_0^{2\pi} \boxed{\int_0^2 -3\,\rho\,\underset{\sim}{(4-\rho^2)}^{1/2}\, d\rho}\; d\vartheta$$

$$\partial_\rho (4-\rho^2)^{3/2} = \frac{3}{2}(4-\rho^2)^{1/2}(-2\rho)$$

$$= 2\pi\cdot\left(-\frac{1}{3}\right)\cdot \qquad = -3(4-\rho^2)\rho$$

$$\Big[(4-\rho^2)^{3/2}\Big]_{\rho=0}^{\rho=2}$$

$$= -\frac{2\pi}{3}\Big[\,0 - \underbrace{4^{3/2}}_{-8}\Big] = +\frac{16}{3}\pi .$$

■

3. $$\int_{\mathbb{R}^2} \underbrace{e^{-(x^2+2y^2)}}_{f(x,y)}\, dx\, dy$$

Sol: f int on $\mathbb{R}^2 \iff \exists \int_{\mathbb{R}^2} |f|$

$$\int_{\mathbb{R}^2} |f| = \int_{\mathbb{R}^2} f$$

$$\overset{RF}{=} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-2y^2}\, dx \quad dy$$

$$e^{-x^2}\, \underbrace{e^{-2y^2}}$$

$$= \int_{-\infty}^{+\infty} e^{-2y^2} \underbrace{\int_{-\infty}^{+\infty} e^{-x^2}\, dx}_{\sqrt{\pi}} \quad dy \overset{(*)}{=}$$

Recall that

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}}\, dx = \int_{-\infty}^{+\infty} e^{-\frac{\left(\frac{x}{\sigma}\right)^2}{2}}\, dx$$

$$\overset{\sigma>0}{=} \quad \sigma \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2}} \, du = \sqrt{2\pi}\cdot\sigma$$

$u = \frac{x}{\sigma}$

$x = \sigma u \qquad dx = \sigma \, du \qquad\qquad = \sqrt{2\pi\sigma^2}$

$$\int_{-\infty}^{+\infty} e^{-x^2} \, dx = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\cdot \overset{=\sigma^2}{\boxed{\frac{1}{2}}}}} \, dx = \sqrt{2\pi \frac{1}{2}}$$

$$= \sqrt{\pi}$$

$$\int_{-\infty}^{+\infty} e^{-\boxed{2}y^2} \, dy = \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\cdot \overset{=\sigma^2}{\boxed{\frac{1}{4}}}}} \, dy = \sqrt{2\pi \frac{1}{4}}$$

$$= \sqrt{\pi/2}$$

$$\overset{(*)}{=} \sqrt{\pi}\cdot\sqrt{\frac{\pi}{2}} = \frac{\pi}{\sqrt{2}} . \quad ■$$

Alt way: $-(x^2+2y^2) = -(\overbrace{x^2+y^2}^{\rho^2} + \overset{\rho^2\cos^2\theta}{y^2})$

$$\int_{\mathbb{R}^2} e^{-x^2-2y^2} \, dx\,dy = \int_{\substack{\rho\geq 0 \\ 0\leq\theta\leq 2\pi}} e^{-\underbrace{\rho^2(1+\cos^2\theta)}} \rho \, d\rho \, d\theta$$

$$\overset{RF}{=} \int_0^{2\pi} \int_0^{+\infty} \underbrace{\rho e^{-(1+\cos^2\theta)\rho^2} d\rho}$$

$$\partial_\rho \, e^{-(1+\cos^2\theta)\rho^2} = e^{-(1+\cos^2\theta)\rho^2} \cdot \left(-(1+\cos^2\theta)\right) 2\rho$$

$$= -\int_0^{2\pi} \frac{1}{2(1+\cos^2\theta)} \int_0^{+\infty} \partial_\rho \, e^{-(1+\cos^2\theta)\rho^2} \, d\rho \, d\theta$$

$$= \ominus \, \frac{1}{2} \int_0^{2\pi} \frac{1}{1+\cos^2\theta} \cdot \Big[\underbrace{e^{-(1+\cos^2\theta)\rho^2}}_{0-1} \Big]_{\rho=0}^{\rho=+\infty} d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{1}{1+\cos^2\theta} \, d\theta .$$

$=$ long.....

$$\int_{\mathbb{R}^2} e^{-\overbrace{(x^2+2y^2)}^{\rho^2}} \, dx \, dy =$$

$$\boxed{x^2+2y^2} = x^2 + (\sqrt{2}\,y)^2 = \rho^2$$

$$\begin{cases} x = \rho\cos\theta \\ \sqrt{2}\,y = \rho\sin\theta \end{cases} \iff \Phi^{-1} \boxed{\begin{cases} x = 1\cdot\rho\cos\theta \\ y = \frac{1}{\sqrt{2}}\rho\sin\theta \end{cases}}$$

$$= \int_{\substack{\rho \geq 0 \\ 0 \leq \theta \leq 2\pi}} e^{-\rho^2} \cdot \underbrace{\overset{?}{\frac{1}{\sqrt{2}}\rho}}_{|\det(\phi^{-1})'(\rho,\theta)|} \, d\rho \, d\theta$$

$$\phi^{-1} \quad \begin{cases} x = \alpha \rho \cos\theta \\ y = \beta \rho \sin\theta \end{cases} \qquad \alpha, \beta > 0$$

$$\det(\phi^{-1})' = \det \begin{bmatrix} \partial_\rho & \partial_\theta \\ \alpha \partial_\rho(\rho\cos\theta) & \alpha \partial_\theta(\rho\sin\theta) \\ \hline \beta \partial_\rho \text{--} & \beta \partial_\rho \text{---} \end{bmatrix}$$

$$= \alpha \beta \underbrace{\det \begin{bmatrix} \partial_\rho(\rho\cos\theta) & \partial_\theta(\rho\cos\theta) \\ \partial_\rho(\rho\sin\theta) & \partial_\theta(\rho\sin\theta) \end{bmatrix}}_{\rho}$$

$$= \alpha\beta\rho$$

$$= \frac{1}{\sqrt{2}} \int_{\rho \geq 0 \;\; 0 \leq \theta \leq 2\pi} \underbrace{\rho e^{-\rho^2}} \, d\rho \, d\theta$$

$$\overset{RF}{=} \frac{1}{\sqrt{2}} \int_0^{+\infty} \rho e^{-\rho^2} \underbrace{\int_0^{2\pi} 1 \, d\theta}_{2\pi} \, d\rho$$

$$= \frac{\cancel{2}\pi}{-\cancel{2}\sqrt{2}} \int_0^{+\infty} -2\rho\, e^{-\rho^2}\, d\rho = -\frac{\pi}{\cancel{2}\sqrt{2}} \left(e^{-\rho^2} \right)_0^{+\infty}$$

$$0 - 1$$

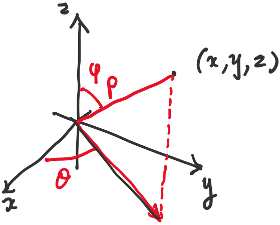
$$\partial_\rho e^{-\rho^2} = e^{-\rho^2} \cdot (-2\rho)$$

$$= \frac{\pi}{\cancel{2}\sqrt{2}} \cdot \blacksquare$$

Integration in spherical and cylindrical coords

We cons. how an int

$$\int_D f(x,y,z)\, dx\, dy\, dz$$



$$\Phi^{-1} \begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$\rho \in [0, +\infty[$

$\boxed{\varphi \in [0, \pi]}$

$\theta \in [0, 2\pi]$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}$$

$$\overset{CV}{=} \int_{D_{\rho,\theta,\varphi}} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \left| \boxed{\rho^2 \sin\varphi} \right| d\rho\, d\theta\, d\varphi$$

$(\Phi(D))$

$$\left| \det (\Phi^{-1})'(\rho, \theta, \varphi) \right|$$

$$\det(\phi^{-1})' = \det \begin{array}{c} \begin{array}{ccc} \partial_\rho & \partial_\theta & \partial_\varphi \end{array} \\ \begin{bmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} \end{array}$$

$$= \cos\varphi \cdot \left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right)$$

$$- 0 \cdot (\qquad\qquad)$$

$$+ (-\rho\sin\varphi)\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right)$$

$$= -\rho^2\cos^2\varphi\sin\varphi - \rho^2\sin^3\varphi$$

$$= -\rho^2\sin\varphi\left(\cos^2\varphi + \sin^2\varphi\right) = -\rho^2\sin\varphi$$

$$\Rightarrow \quad \left|\det(\phi^{-1})'(\rho,\theta,\varphi)\right| = \left|-\rho^2\sin\varphi\right|$$

$$= \boxed{\rho^2\sin\varphi}$$

Ex. 5.4.5 Volume of a sphere of radius r

$$\lambda_3\left(\{(x,y,z)\in\mathbb{R}^3 : \; x^2+y^2+z^2\le r^2\}\right)$$

$$= \int_{\underbrace{x^2+y^2+z^2\le r^2}_{}} \textcircled{1} \; dx\,dy\,dz$$

$$\underset{\text{spher. coords}}{=} \int_{\substack{\boxed{\rho^2\le r^2}\\ \boxed{0\le\theta\le 2\pi}\\ 0\le\varphi\le\pi}} \boxed{1 \; \boxed{\rho^2\sin\varphi}} \; \underbrace{d\rho\,d\theta\,d\varphi}_{}$$

$$\overset{RF}{=} \int_{\substack{0\le\rho\le r\\ 0\le\varphi\le\pi}} \left(\rho^2\sin\varphi \; \underset{\substack{\shortparallel\\ 2\pi}}{\int_0^{2\pi} 1\,d\theta}\right) d\rho\,d\varphi$$

$$= 2\pi \int_{\substack{0\le\rho\le r\\ 0\le\varphi\le\pi}} \underbrace{\rho^2\sin\varphi}_{} \; d\rho\,d\varphi$$

$$\overset{RF}{=} \underbrace{2\pi}_{} \int_0^r \rho^2 \overset{\overset{+2}{/\!/}}{\underbrace{\int_0^\pi \sin\varphi\,d\varphi}_{}} \; d\rho$$

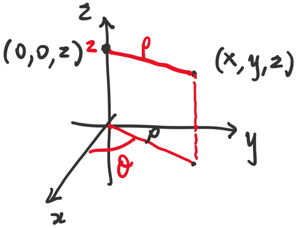
$$-\cos\varphi\,\Big|_0^\pi = -(-1-1)$$

$$= +2$$

$$= 4\pi \int_0^r \rho^2 \, d\rho = 4\pi \left. \frac{\rho^3}{3} \right|_0^r$$

$$= \frac{4\pi}{3} r^3 . \quad \blacksquare$$

Cylindrical coords



$$\rho = \| (x,y,z) - (0,0,z) \|$$

$$= \| (x,y,0) \| = \sqrt{x^2+y^2}$$

$$\boxed{x^2 + y^2 = \rho^2}$$

$$\phi^{-1} \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \\ z = \boxed{z} \end{cases}$$

$$\int_D f(x,y,z)\,dx\,dy\,dz \underset{\text{cyl. coords}}{=} \int_{\underbrace{D_{\rho,\theta,z}}_{\phi(D)}} f(\rho\cos\theta, \rho\sin\theta, z) \underset{|\det(\phi^{-1})'|}{\rho} \, d\rho\, d\theta\, dz$$

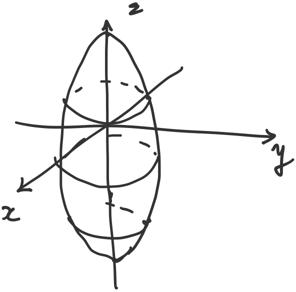
$$\det(\phi^{-1})' = \det \begin{array}{c} \begin{array}{ccc} \partial_\rho & \partial_\theta & \partial_z \end{array} \\ \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$= \rho$

Ex. Vol of a rugby ball

$$D = \left\{ (x,y,z) : \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1 \right\}$$

$a, b > 0$



Sol:

$$\lambda_3(D) = \int_D 1 \; dx\,dy\,dz$$

$$\underset{\text{cyl. coords}}{=} \int_{\substack{\frac{\rho^2}{a^2} + \frac{z^2}{b^2} \leq 1 \\ \boxed{0 \leq \theta \leq 2\pi}}} \boxed{1 \cdot \rho} \; d\rho\, d\theta\, dz$$

$$(x,y,z) \in D \iff \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1$$

$$x^2+y^2 = \rho^2 \quad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad \boxed{\frac{\rho^2}{a^2} + \frac{z^2}{b^2} \leq 1 \quad 0 \leq \theta \leq 2\pi}$$

$$\overset{RF}{=} \int_{\frac{\rho^2}{a^2}+\frac{z^2}{b^2}\le 1} \rho \left(\underbrace{\int_0^{2\pi} 1 \, d\theta}_{=2\pi} \right) d\rho\, dz$$

$$= 2\pi \int_{\boxed{\frac{\rho^2}{a^2}+\frac{z^2}{b^2}\le 1}} \rho \; d\rho\, dz$$

$$\overset{RF}{=} 2\pi \int \int_{\cancel{-\sqrt{}}\;0\,\checkmark}^{+\sqrt{}} \rho\, d\rho \quad dz$$

$$\frac{\rho^2}{b^2} + \frac{z^2}{b^2} \le 1 \Leftrightarrow \frac{\rho^2}{\cancel{a^2}} \le \underbrace{\left(1-\frac{z^2}{b^2}\right)}_{} a^2$$

$$\Leftrightarrow -\sqrt{} \le \rho \le \sqrt{}$$

$$+ \; \rho \ge 0$$

$$\Leftrightarrow \quad 0 \le \rho \le \sqrt{}$$

$$= 2\pi \int_{-b}^{b} \left(\int_0^{\sqrt{a^2\left(1-\frac{z^2}{b^2}\right)}} \rho\, d\rho \right) dz$$

$$1 - \frac{z^2}{b^2} \geq 0 \iff z^2 \leq b^2 \iff -b \leq z \leq b$$

$$= \cancel{2}\pi \int_{-b}^{b} \frac{\rho^2}{\cancel{2}} \Big|_0^{\sqrt{}} \quad dz$$

$$= \pi \int_{-b}^{b} a^2\left(1 - \frac{z^2}{b^2}\right) - 0 \quad dz$$

$$= \pi a^2 \left(2b - \frac{1}{b^2}\boxed{\int_{-b}^{b} z^2 \, dz}\right)$$

$$\parallel$$

$$\frac{z^3}{3}\Big|_{-b}^{b} = \frac{1}{3}\left(b^3 - (-b^3)\right)$$

$$= \frac{2}{3} b^3$$

$$= \pi a^2 \left(2b - \frac{1}{\cancel{b^2}} \cdot \frac{2}{3} b^{\cancel{3}}\right)$$

$$= \pi a^2 b \frac{4}{3}. \quad ■$$

Do 5.7.5 # 4, 5, 6

5.7.4 # 6, 8, 1, 7.

↑ ↑ ↑

figures