NUMERICAL AND MATHEMATICAL METHODS

August 30, 2021

EXERCISE 1. Please, download the file data_black_holes.txt, which contains information about the evolution of 147114 simulated black holes. The first three lines are the header of the file: please, skip them when reading the file.

After the three-line header, columns 3 and 19 [starting from zero] are the mass of the black hole ($m_{\rm BH}$ in solar masses, $M_\odot$) and the age of its stellar progenitor when it underwent core collapse ($t_{\rm SN}$ in Myr), respectively. In this file, columns are separated by spaces.

Write a python script that

A. reads columns 3 and 19 of this file;

B. calculates the $\log_{10}$ values of $m_{\rm BH}$ and $t_{\rm SN}$;

C. does a scatter plot of $t_{\rm SN}$ versus $m_{\rm BH}$ (plot their logarithmic values and, please, remember to add the axis labels and corresponding physical units);

D. does a linear fit of the $\log_{10}$ values of $m_{\rm BH}$ and $t_{\rm SN}$ using the function scipy.optimize.curve_fit();

E. prints the values of the best fitting parameters and their corresponding deviation;

F. over-plots the best-fitting line onto the scatter plot described at point **C.**.

The plot for points C. and F. do not need to be saved to a file. Just add the command plt.show() inside the script.

Please upload the script developed to address points A., B., C., D., E. and F. as a unique script named exercise1.py.

EXERCISE 2. The following equation describes a three-dimensional Bernoulli lemniscate in Cartesian coordinates:

$$\left(x^2 + y^2 + z^2\right)^2 = a^2 \left(x^2 - y^2 - z^2\right), \tag{1}$$

where a is a parameter. Let us assume $a = 1$.

A. Write a script that uses the rejection method to generate 10^5 points uniformly distributed **inside** the region defined by the three-dimensional lemniscate, i.e. satysfying the following criterion

$$\left(x^2 + y^2 + z^2\right)^2 \leq a^2 \left(x^2 - y^2 - z^2\right). \tag{2}$$

Suggestion: to generate the random numbers, start from a rectangular prism centered in zero with sides of length $(2, 1, 1)$ *along the* x, y *and* z *axis, respectively. The lemniscate with* $a = 1$ *fits nicely inside this simple prism.*

B. Plot the resulting random points with a three-dimensional scatter plot, in which the axes indicate the x, y and z coordinates.

C. Now use the simplest possible Monte Carlo technique to solve integrals in three dimensions and calculate the volume of the three-dimensional lemniscate.

The plot for point B. does not need to be saved to a file. Just add the command plt.show() inside the script.

Please upload the script developed to address points A., B. and C. as a unique script named exercise2.py.

EXERCISE 3. Planck's radiation law tells us that the intensity of radiation per unit area and per unit wavelength λ from a black body at temperature T is

$$I(\lambda) = \frac{2\,\pi\,h\,c^2\,\lambda^{-5}}{\exp\left(\frac{h\,c}{\lambda\,\kappa_{\mathrm{B}}\,T}\right) - 1}, \tag{3}$$

where h is Planck's constant, c is the speed of light and κ_{B} is Boltzmann's constant. By differentiating the above equation, it can be shown that the wavelength $\lambda_{\max}$ at which the emitted radiation is maximum is the solution of the following equation

$$5\,\exp\left(-x\right) + x - 5 = 0, \tag{4}$$

where $x \equiv \frac{h\,c}{\lambda_{\max}\,\kappa_{\mathrm{B}}\,T}$.

A. Write a script to solve eq. (4) to an accuracy of $\epsilon = 10^{-5}$ with the bisection method.

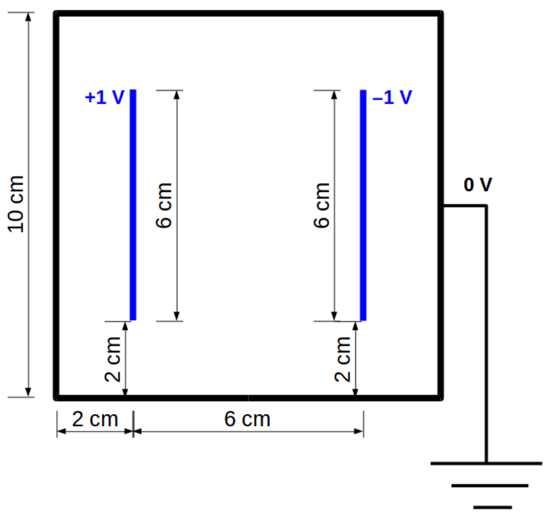
B. Inside the script you wrote to address point **A.**, print the value of the constant b of the *Wien displacement law*:

$$\lambda_{\max} = \frac{b}{T}, \tag{5}$$

where $b = h\,c/(\kappa_{\mathrm{B}}\,x)$.

Please upload the scripts developed to address points A. and B. as a single script, named exercise3.py.

EXERCISE 4. Consider the **two-dimensional** electronic capacitor shown in the Figure, consisting of two flat metal plates (shown in blue in the Figure) enclosed in a **square** metal box with **10 cm long side**:



The plates are located as in the Figure and are 6 cm long. One of them is kept at a voltage of +1 V and the other at −1 V. The walls of the box are at 0 V. Assuming the size of the plates is negligible (i.e., it is the same as your numerical resolution),

A. write a script to solve the two-dimensional Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{6}$$

Suggestions: This is a static boundary problem. The method of finite differences is good for it. A grid with 100×100 *squared cells of side* $a = 0.1$ *cm will be perfect to solve this problem. A tolerance of* 10^{-3} *is fine.*

B. Plot the solution with a two-dimensional regular raster (i.e., use the matplotlib.pyplot.imshow() function).

C. Try to solve this problem again with overrelaxation.

Please upload the script developed to address points A., B., and C. as a unique script named exercise4.py.