Thursday, 21 December 2023 08:35

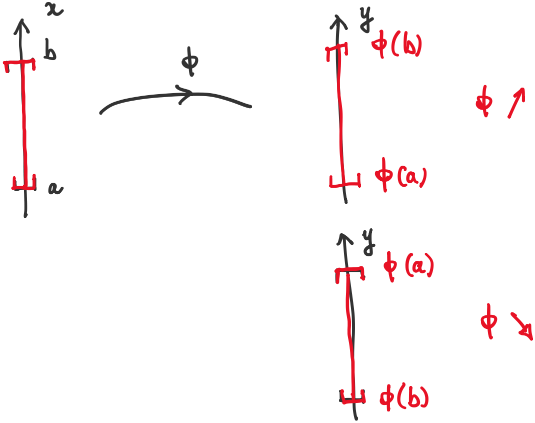
Change of vars formula

We remind that

$$\int_a^b f(x)\,dx \underset{y=\phi(x)}{=} \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy$$

$$dx = (\phi^{-1})'(y)\,dy \qquad \boxed{x = \phi^{-1}(y)}$$

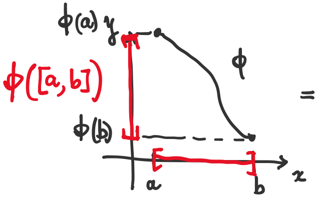
$$(\phi \in \mathcal{C}^1,\ \exists\ \phi^{-1} \in \mathcal{C}^1)$$



If $\phi \searrow$

$\phi(b)$

$$\int_a^b f(x)\,dx = \int_{\phi(a)} f(\phi^{-1}(y))\,(\phi^{-1})'(y)\,dy$$



$$= \cancel{-}\int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y))\,\boxed{-(\phi^{-1})'(y)}\,dy$$

$$\parallel$$

$$|(\phi^{-1})'(y)|$$

$$\phi \searrow \iff \phi^{-1} \searrow$$

$$= \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y))\,|(\phi^{-1})'(y)|\,dy$$

So, if $\phi \searrow$

$$\int_a^b f(x)\,dx = \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y))\,|(\phi^{-1})'(y)|\,dy$$

$$\parallel$$

$$\int_{[a,b]} f(x)\,dx = \int_{[\phi(b),\,\phi(a)]} f(\phi^{-1}(y))\,|(\phi^{-1})'(y)|\,dy$$

$$[\phi(b),\phi(a)] = \phi([a,b])$$

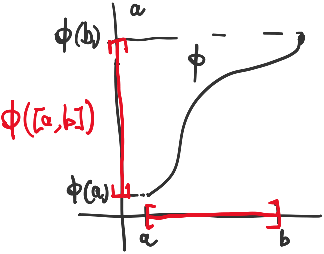
that is, for $\phi \searrow$

$$\int f(x)\,dx = \int f(\phi^{-1}(y))\,|(\phi^{-1})'(y)|\,dy$$

$[a,b]$ $\qquad\qquad$ $\phi([a,b])$

If ϕ ↗

$$\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))\,(\phi^{-1})'(y)\,dy$$



$$\phi \nearrow \iff \phi^{-1} \nearrow$$

$$(\phi^{-1})' > 0$$

that is

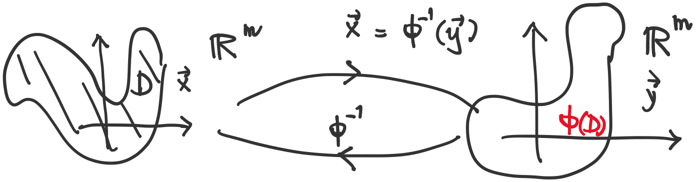
$$\int_{[a,b]} f(x)\,dx = \int_{\phi([a,b])} f(\phi^{-1}(y))\,\left|(\phi^{-1})'(y)\right|\,dy$$

Conclusion: whatever is ϕ ($\nearrow$ or $\searrow$)

$$\int_{[a,b]} f(x)\,dx = \int_{\phi([a,b])} f(\phi^{-1}(y))\,\left|(\phi^{-1})'(y)\right|\,dy.$$

Let's now consider a multiple int

$$\int_D f(\vec{x})\,d\vec{x} \underset{\vec{y}=\phi(\vec{x})}{=} \qquad \phi, \phi^{-1} \text{ diff.}$$



$$= \int_{\phi(D)} f(\phi^{-1}(\vec{y})) \, | \, ? \, | \, d\vec{y}$$

$(\phi^{-1})'$ is the jac. matrix, an $m \times m$ matrix

Thm: $D \subset \mathbb{R}^m$, $\phi : D \longrightarrow \boxed{\phi(D)} \subset \mathbb{R}^m$

ϕ invertible, ϕ, ϕ^{-1} diff on their domains

Let $f = f(\vec{x}) : D \longrightarrow \mathbb{R}$. Then

$$\exists \int_D f(\vec{x}) \, dx \Leftrightarrow \exists \int_{\phi(D)} f(\phi^{-1}(\vec{y})) \, |\det(\phi^{-1})'(\vec{y})| \, d\vec{y}$$

and, in this case,

$$\int_D f(\vec{x}) \, d\vec{x} \underset{\substack{\vec{y} = \phi(\vec{x}) \\ \vec{x} = \phi^{-1}(\vec{y})}}{=} \int_{\phi(D)} f(\phi^{-1}(\vec{y})) \, |\det(\phi^{-1})'(\vec{y})| \, d\vec{y}$$

$|\det(\phi^{-1})'(\vec{y})|$ is called jacobian of ϕ^{-1}.

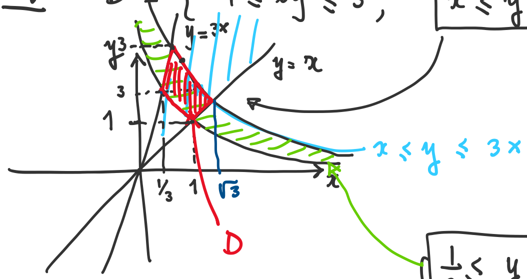
Ex 5.7.6

1. Compute $\int_D xy\,dx\,dy$

$D = \{(x,y)\in\mathbb{R}^2 : 1 \le xy \le 3,\ x \le y \le 3x\}$

through change of vars $\boxed{\begin{array}{l} u = xy \\ v = y/x \end{array}}$.

Sol: $D = \{ 1 \le xy \le 3,\ \boxed{x \le y \le 3x}\}$



$$1 \le xy \le 3 \iff \begin{cases} \boxed{\frac{1}{x} \le y \le \frac{3}{x} \quad x>0} \\ \emptyset & x=0 \\ \frac{3}{x} \le y \le 1/x & x<0 \end{cases}$$

D is closed and bounded $\left(\frac{1}{3} \le x \le \sqrt{3},\ 1 \le y \le 3\right)$

$f \in \mathcal{C}(D) \Rightarrow \exists \int_D f.$

To compute the int we apply change of vars formula

$$\int_D f(x,y)\,dx\,dy = \int_{\boxed{\phi(D)}} f(\phi^{-1}(u,v))\,\left|\det(\phi^{-1})'(u,v)\right|\,du\,dv$$

$$\phi\left\{\boxed{\begin{matrix}u\\ \\ v\end{matrix}}\begin{matrix}=\\ \\ =\end{matrix}\boxed{\begin{matrix}xy \leftarrow\\ \\ y/x\end{matrix}}\right. \qquad \phi^{-1}\begin{cases} x = \\ y = \end{cases} \text{functs of } (u,v)$$

$$\Downarrow$$

$$uv = xy\cdot\frac{y}{x} = y^2 \qquad \begin{cases} y^2 = \boxed{uv} > 0 \\ x = \dfrac{u}{y} \end{cases}$$

$$u = xy$$

$$\phi\begin{cases} u = xy \\ v = y/x \end{cases} \qquad (x,y)\in D \Rightarrow \boxed{x,y>0}$$

$$\Rightarrow \begin{matrix} u = xy > 0 \\ v = y/x > 0 \end{matrix}$$

$$\Rightarrow uv > 0$$

and $y^2 = uv \iff \boxed{y = \sqrt{uv}}$ $\quad \xcancel{y = -\sqrt{uv}}$

$$\begin{cases} y = \sqrt{uv} \\ x = \dfrac{u}{y} = \dfrac{u}{\sqrt{uv}} = \sqrt{\dfrac{u}{v}}. \end{cases}$$

So, for $(x,y)\in D$

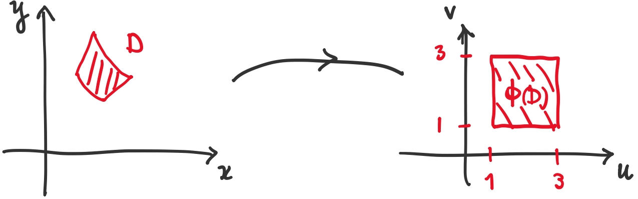
$$\phi \begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases} \iff \phi^{-1} \begin{cases} x = \boxed{\sqrt{\dfrac{u}{v}}} = u^{1/2} v^{-1/2} \\ y = \sqrt{uv} = \underline{u^{1/2} v^{1/2}} \end{cases}$$

We compute now the jacobian

$$\det (\phi^{-1})' = \det \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \partial_u & \partial_v \\ \end{array}$$

$$\det (\phi^{-1})' = \det \begin{bmatrix} \frac{1}{2} u^{-1/2} v^{-1/2} & -\frac{1}{2} u^{1/2} v^{-3/2} \\ \frac{1}{2} u^{-1/2} v^{1/2} & \frac{1}{2} u^{1/2} v^{-1/2} \end{bmatrix}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \left(\underbrace{v^{-1} + v^{-1}}_{2/v} \right) = \frac{1}{2\cancel{4}v}$$



$$D = \{ (x,y) : \underbrace{1 \leq xy \leq 3}_{1 \leq u \leq 3}, \quad x \leq y \leq 3x \}$$

Since $x > 0$ for $(x,y) \in D$

$$x \leq y \leq 3x \iff 1 \leq \frac{y}{x} \leq 3$$

$$1 \leq v \leq 3$$

$$\phi(D) = \{(u,v) : \quad 1 \le u \le 3, \quad 1 \le v \le 3\}$$

$$\int_D \boxed{xy}\, dx\, dy = \int_{\substack{1 \le u \le 3 \\ 1 \le v \le 3}} \overset{f(\phi^{-1}(u,v))}{u} \cdot \overset{|\det(\phi^{-1})'(u,v)|}{\frac{1}{2\cancel{4}v}}\, du\, dv$$

$$= \frac{1}{\cancel{4}2} \int_{\substack{\boxed{1 \le u \le 3} \\ 1 \le v \le 3}} \frac{u}{v}\, du\, dv$$

$$\overset{RF}{=} \frac{1}{\cancel{4}2} \int_1^3 \frac{1}{v} \overbrace{\left(\int_1^3 \frac{u}{\cancel{v}}\, du \right)}^{= 4} dv$$

$$\frac{u^2}{2}\Big|_1^3 = \frac{1}{2}(9-1) = 4$$

$$= 2 \int_1^3 \frac{1}{v}\, dv = 2\log|v| \Big|_{v=1}^{v=3} = 2\log 3 \quad \blacksquare$$

Rmk: (A shortcut for jacobian)

$$\det(\phi^{-1})'$$

Since $\phi(\underbrace{\phi^{-1}(\vec{y})}) = \vec{y} \quad \forall \vec{y} \in \phi(D)$

$\Downarrow$ diff.

$$\boxed{\phi'(\phi^{-1}(\vec{y}))\cdot(\phi^{-1})'(\vec{y})} = \mathbb{I}_m \quad (*)$$

$$\vec{y} \xrightarrow[\text{Id}]{} \vec{y} = (\underset{\uparrow}{\boxed{y_1}} \dots y_m) \quad \text{Id}' = \begin{array}{c} \partial_{y_1} \quad \partial_{y_2} \cdots\cdots \partial_{y_m} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \dots & 0 \\ 0 & \_\_\_\_ & & 1 \end{bmatrix} \end{array} = \mathbb{I}_m$$

$$(\phi^{-1})'(\vec{y}) = \left[\phi'(\phi^{-1}(\vec{y}))\right]^{-1}$$

$$\left((f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \right)$$

Applying det to both sides of (*) we get

$$\det\left[\phi'(\phi^{-1}(\vec{y}))\ (\phi^{-1})'(\vec{y})\right] = \det \mathbb{I}_m = 1$$

and since $\det(AB) = \det A \cdot \det B$

A, B are $m \times m$ matrices

we obtain

$$\det(\phi^{-1})'(\vec{y}) = \overline{\det\left(\phi'(\underbrace{\phi^{-1}(\vec{y})}_{\vec{x}})\right)}$$

In the case of the Exercise:

$$\phi \begin{cases} u = \boxed{xy} \\ v = \boxed{y/x} \end{cases}$$

$$\det \phi'(x,y) = \det \begin{array}{c} \begin{array}{cc} \partial_x & \partial_y \end{array} \\ \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix} \end{array}$$

$$= \frac{y}{x} + \cancel{x}\frac{y}{x^{\cancel{2}}} = 2\boxed{\frac{y}{x}}$$

$$\det \phi'(\phi^{-1}(u,v)) = 2v$$

$$\Rightarrow \quad |\det(\phi^{-1})'(u,v)| = \left|\frac{1}{2v}\right| = \frac{1}{2v} \text{ on}$$

$$\phi(D).$$

$\blacksquare$

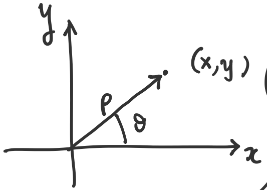
Do 5.7.6 # 2, 3

It is interesting to introduce some special change of vars.

Polar coords.

$$\int f(x,y)\,dx\,dy =$$

D



$$\phi^{-1}\begin{cases} x = \boxed{\rho\cos\theta} \\ y = \boxed{\rho\sin\theta}\end{cases}$$

$$\phi\begin{cases} \rho = \sqrt{x^2+y^2} \\ \theta = \text{complicate}\end{cases}$$

jacobian : $|\det(\phi^{-1})'(\rho,\theta)| =$

$$= \left|\det \underbrace{\left[\begin{array}{cc} \partial_\rho & \partial_\theta \\ \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{array}\right]\right|}$$

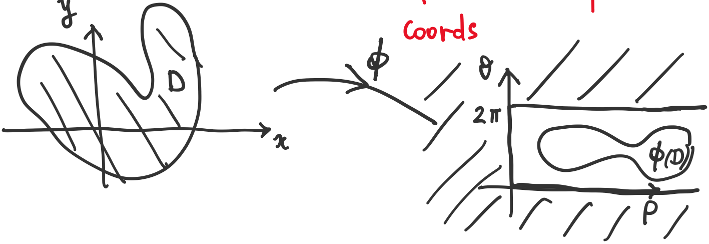
$$|\rho\cos^2\theta + \rho\sin^2\theta| = |\rho| = \rho$$

$$\boxed{|\det(\phi^{-1})'(\rho,\theta)| = \rho}$$

So

!o

$$\int_D f(x,y)\,dx\,dy = \int_{\boxed{\phi(D)}} f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$



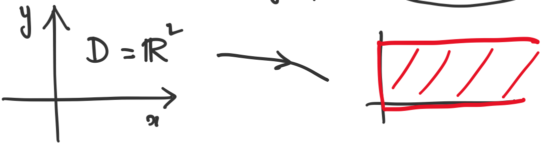
D represented in polar coords

Ex. 5.4.3

Compute $$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\,dxdy.$$

Sol: We apply the ch. of vars in pol. coords:

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}} \underset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \boxed{\int_{\substack{\rho\geq 0\\ 0\leq\theta\leq 2\pi}} e^{-\rho}\cdot\rho \;d\rho d\theta}$$



$$\overset{RF}{=} \int_0^{+\infty}\int_0^{2\pi} \rho e^{-\rho}\,d\theta\,d\rho$$

$$= 2\pi\int_0^{+\infty} \rho\underset{\shortparallel}{e}^{-\rho}\,d\rho$$

$$\partial_\rho(-e^{-\rho})$$

$$= 2\pi \left[\underset{0}{-\rho e^{-\rho} \Big|_0^{+\infty}} + \int_0^{+\infty} + e^{-\rho}\, d\rho \right]$$

$$= 2\pi \int_0^{+\infty} e^{-\rho}\, d\rho = 2\pi \underset{0-(-1)}{\left[-e^{-\rho}\right]_0^{+\infty}}$$

$$= 2\pi . \quad \blacksquare$$

Ex:

$$\int_{\mathbb{R}} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Sol:

$$I = \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx$$

$$I^2 = I \cdot I = \int_{-\infty}^{+\infty} e^{-x^2/2} dx \int_{-\infty}^{+\infty} e^{-y^2/2} dy$$

$$= \int_{-\infty}^{+\infty} \boxed{e^{-x^2/2}} \int_{-\infty}^{+\infty} e^{-y^2/2}\, dy\, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

$$= \int_{\boxed{\mathbb{R}^2}} e^{-\frac{x^2+y^2}{2}}\, dx\, dy$$

$$\underset{\text{pol coords}}{=} \int_{\substack{\rho \geq 0 \\ 0 \leq \vartheta \leq 2\pi}} e^{-\rho^2/2} \rho \, d\rho \, d\vartheta$$

$$\overset{RF}{=} \int_0^{+\infty} \int_0^{2\pi} \boxed{e^{-\rho^2/2} \rho} \, d\vartheta \, d\rho$$

$$= -2\pi \int_0^{+\infty} - e^{-\rho^2/2} \rho \, d\rho$$

$$\partial_\rho \underbrace{e^{-\rho^2/2}} = e^{-\rho^2/2} \cdot (-\rho)$$

$$= -2\pi \left[e^{-\rho^2/2} \right]_{\rho=0}^{\rho=+\infty}$$

$$0 - 1$$

$$= 2\pi \qquad \Longrightarrow \qquad I^2 = 2\pi$$

$$\Longrightarrow \qquad I = \cancel{\pm}\sqrt{2\pi}$$

■

Do Ex 5.7.3

5.7.5 # 1,2,3.