Wednesday, 20 December 2023 08:34

Ex 5.7.1

# 4 $$\int_{[0,1]\times[2,4]} \frac{1}{(x-y)^2}\,dxdy.$$

$\underline{\text{Sol:}}$ $D = \{(x,y) : \underline{0 \leq x \leq 1},\ \underline{2 \leq y \leq 4}\}$

D is clearly closed & bded,

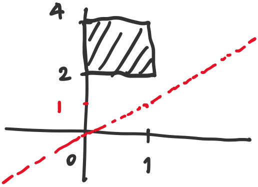
$f \in \mathcal{C}(\mathbb{R}^2 \setminus \{y=x\})$

$\Downarrow$

$f \in \mathcal{C}(D)$

$\Downarrow$

$\exists \int_D f$



Let's compute $$\int_D f \overset{RF}{=} \int_0^1 \int_2^4 \underbrace{\frac{1}{(x-y)^2}} dy\, dx$$

$$= \int_0^1 \left[\ominus \frac{1}{y-x} \right]_{y=2}^{y=4} dx$$

$$\frac{1}{(x-y)^2} = \frac{1}{(y-x)^2} = \partial_y \frac{-1}{y-x}$$

$$\partial_y \frac{1}{y-x} = \partial_y (y-x)^{-1} = (-1)\cdot(y-x)^{-2}$$

$$= -\int_0^1 \frac{1}{4-x} - \frac{1}{2-x} \, dx$$

$$= \int_0^1 \frac{1}{x-4} + \frac{1}{x-2} \, dx$$

$$= \log|x-4| \Big|_0^1 + \log|x-2| \Big|_0^1$$

$$= \log 3 - \log 4 + \cancel{\log 1} - \log 2$$

$\blacksquare$

5. $$\int_{1 \leq x \leq 2,\ \frac{1}{x} \leq y \leq x} \frac{x}{y} \, dx\, dy$$

Sol: Integrability: $D = \left\{ (x,y) : \begin{array}{l} 1 \leq x \leq 2 \\ \frac{1}{x} \leq y \leq x \end{array} \right\}$

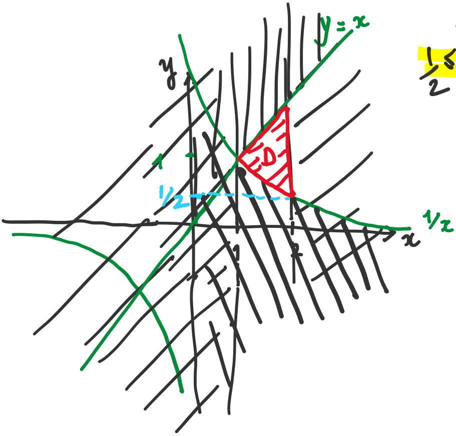
D is clearly closed; it is also bdd because

$$(x,y)\in D \Rightarrow 1 \le \boxed{x} \le 2$$

$$\frac{1}{2} \le \boxed{\frac{1}{x} \le y \le x} \le 2$$

$$\Downarrow$$

$$\frac{1}{2} \le y \le 2$$

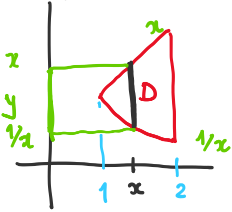


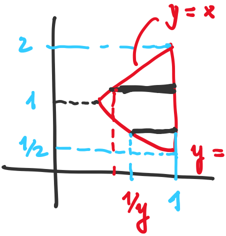
$$f\in \mathcal{C}(\mathbb{R}^2 \setminus \{y=0\}) \Rightarrow f\in \mathcal{C}(D)$$

$$\Rightarrow \exists \int_D f.$$

To compute the int we apply RF:

$$\int_D f = \boxed{\int_1^2 \int_{1/x}^{x} \frac{x}{y}\, dy \quad dx}$$





$1/2 \le y \le 1$ $\boxed{\frac{1}{y} \le x \le 1}$

$1 \le y \le 2$ $\boxed{y \le x \le 1}$

$y = 1/x \iff x = 1/y$

$$= \int_{y:\, D^y \neq \emptyset} \int_{\enclose{circle}{D^y}} \frac{x}{y}\, dx \quad dy \swarrow$$

$$= \int_{1/2}^{1} \int_{1/y}^{1} \frac{x}{y}\, dx\, dy + \int_{1}^{2} \int_{y}^{1} \frac{x}{y}\, dx\, dy$$

We proceed with first formula

$$\int_D f = \int_1^2 \int_{1/x}^{x} \frac{\enclose{circle}{x}}{y}\, dy \quad dx$$

$$= \int_1^2 x \underset{\log|y|\,\Big|_{y=1/x}^{y=x}}{\underset{\parallel}{\int_{1/x}^{x} \frac{1}{y}\, dy}} \quad dx$$

$$= \int_1^2 x \left(\log x - \underset{\parallel}{\log \frac{1}{x}} \right) dx$$

$$\partial_x \frac{x^2}{2} \qquad + \log x$$

$$= 2 \int_1^2 \overset{//}{x} \log x \; dx$$

$$\overset{\text{by parts}}{=} 2 \left[\frac{x^2}{2} \log x \Big|_{x=1}^{x=2} - \int_1^2 \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}} dx \right]$$

$$= 2 \left[\frac{1}{2} (4 \log 2) - \frac{1}{4} \underset{4-1}{\left[\frac{x^2}{\cancel{2}} \right]_1^2} \right]$$

$$= 4 \log 2 - \frac{3}{2} . \qquad ■$$

Similarly to double integrals

$$\int_D f(x,y) \, dx \, dy$$

triple integrals

$$\int_D f(x,y,z) \, dx \, dy \, dz$$

and more general multiple integrals

$$\int_D f(x_1, x_2, \dots x_m)\, dx_1 — dx_m$$

are defined with similar properties.

Also reduction formula holds. Let's see how it works. For double integrals

$$\int_D f(x,y)\,dx\,dy = \int_{x : D_x \neq \phi} \int_{D_x} f(x,y)\,dy \quad dx$$

$$D_x = \{ y : (x,y) \in D \}$$

$$= \int_{y : D^y \neq \phi} \int_{D^y} f(x,y)\,dx \quad dy$$

$$D^y = \{ x : (x,y) \in D \}$$

In general we have :

Thm: f integrable on D. $(f = f(x_1, \dots x_m))$

Then

$$\int_D f(\underbrace{x_1, \dots}_{\vec{z}} \,|\, \underbrace{\dots x_m}_{\vec{y}})\, dx_1 \dots dx_m =$$

$$\vec{x} = (x_1, \dots, x_k)$$

$$\vec{y} = (\underbrace{x_{k+1}}, \cdots, \underbrace{x_m})$$

$$= \int_{\vec{x} : D_{\vec{x}} \neq \phi} \int_{D_{\vec{x}}} f(\vec{x}, \vec{y}) \underset{dx_{k+1} \cdots dx_m}{d\vec{y}} \quad \overset{dx_1 \cdots dx_k}{\overset{\nearrow}{d\vec{x}}}$$

$$D_{\vec{x}} = \{ \vec{y} : (\vec{x}, \vec{y}) \in D \}$$

$$= \int_{\vec{y} : D^{\vec{y}} \neq \phi} \left(\int_{D^{\vec{y}}} f(\vec{x}, \vec{y}) \, d\vec{x} \right) d\vec{y}$$

$$D^{\vec{y}} = \{ \vec{x} : (\vec{x}, \vec{y}) \in D \}$$

For $m = 3$

$$\int_D f(x, y \,|\, \overset{\downarrow}{z}) \, dx \, dy \, dz =$$

$$= \int_{(x,y) : D_{(x,y)} \neq \phi} \left(\int_{D_{(x,y)}} f(x, y, z) \, dz \right) dx \, dy$$

$$D_{(x,y)} = \{ z \ : \ (x,y,z) \in D \}$$

$$= \int_{z : D^z \neq \phi} \left(\int_{D^z} f(\boxed{x,y}, z) \, dx \, dy \right) dz$$

$$D^z = \{ (x,y) \ : \ (x,y,z) \in D \}$$

$$\int_D f(x \,|\, y, z) \, dx \, dy \, dz =$$

$$= \int_{x : D_x \neq \phi} \left(\int_{D_x} f(x, y, z) \, dy \, dz \right) dx$$

$$= \int_{(y,z) : D^{(y,z)} \neq \phi} \left(\int_{D^{(y,z)}} f(x, \underline{y, z}) \, dx \right) dy \, dz$$

$$\int_D f(x, y, z) = \int_{\substack{(x,z) : \\ D_{(x,z)} \neq \phi}} \int_{D_{(x,z)}} f(x,y,z) \, dy \quad dx \, dz$$

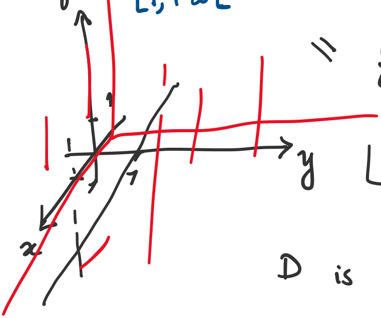
$$D_{(x,z)} = \{ y \ : \ (x,y,z) \in D$$

$$= \int_{y : D^y \neq \emptyset} \left(\int_{D^y} f(x,y,z)\, dx\, dz \right) dy$$

$$D^y \rightarrow \{(x,z) : (x,y,z) \in D\}$$

Ex 5.7.2.

1. $$\int_{[1,+\infty[^3} y^3 z^8 e^{-xy^2z^3}\, dx\, dy\, dz$$



$$= \underbrace{\left\{ (x,y,z) : \begin{matrix} x \geq 1 \\ y \geq 1 \\ z \geq 1 \end{matrix} \right\}}_{D}$$

D is closed and unbdd.

To verify integrability we use the int test coming from RF: we notice that

$$f(x,y,z) = y^3 z^8 e^{-xy^2z^3} \geq 0$$

on D

To check int, we compute an iterated int for f

$$\boxed{\int_{y\geq 1, z\geq 1} \left(\int_{x \geq 1} f \, dx \right) dy\, dz}$$

$$= \int_{y\geq 1,\, z\geq 1} \left(\int_1^{+\infty} y^3 z^8 e^{-xy^2z^3} dx \right) dy\,dz$$

$$= -\int_{y\geq 1,\, z\geq 1} y^{\cancel{3}} z^{\cancel{8}5} \int_1^{+\infty} -y^2z^3 \boxed{e^{-xy^2z^3}} dx \quad dy\,dz$$

$$\partial_x e^{-xy^2z^3} = -y^2z^3 e^{-xy^2z^3}$$

$$= -\int_{\underline{y\geq 1,\, z\geq 1}} y z^5 \left[e^{-xy^2z^3} \right]_{x=1}^{x=+\infty} dy\,dz$$

$$\Bigg[\lim_{x\to+\infty} e^{-x\underbrace{y^2z^3}_{\geq 1}} - e^{-y^2z^3}$$

$-\infty$

$\downarrow$ 0

$$= + \int y z^5 \left(+ e^{-y^2z^3} \right) dy\,dz$$

$y \geq 1,\ z \geq 1$

$$\overset{RF}{=} -\frac{1}{2}\int_1^{+\infty} z^2 \int_1^{+\infty} -2yz^{\overset{3}{\cancel{5}}} e^{-y^2z^3}\, dy \quad dz$$

$$\partial_y\ e^{-y^2z^3} = e^{-y^2z^3}\left(-2yz^3\right)$$

$$= -\frac{1}{2}\int_1^{+\infty} z^2 \left[\underset{0\,-\,e^{-z^3}}{\overset{\overset{1}{\wedge}}{e^{-y^2z^3}}}\right]_{y=1}^{y=+\infty} dz$$

$$= \frac{1}{-3\cdot 2}\int_1^{+\infty} -3z^2 e^{-z^3}\, dz$$

$$\partial_z\ e^{-z^3} = e^{-z^3}\left(-3z^2\right)$$

$$= -\frac{1}{6}\left[\underset{0\,-\,e^{-1}}{e^{-z^3}}\right]_{z=1}^{z=+\infty} = \frac{1}{6e}.$$

$$\Rightarrow \quad \int \Big(\int \underset{|f|}{\underset{\shortparallel\ f \geq 0}{f}} \, dx \Big) \, dy \, dz = \frac{1}{6e}$$

$$\Rightarrow \exists \int_D |f| < +\infty \quad \Rightarrow \exists \int_D f$$

$$\Rightarrow \quad \int_D f \overset{RF}{=} \quad = \frac{1}{6e} . \quad \blacksquare$$

Def: (measure of a set D in $\mathbb{R}^m$)

$$\lambda_m(D) := \int_{\mathbb{R}^m} \boxed{1_D} \equiv \int_D 1 .$$

provided 1_D is integrable.

Rmk: There're set for which $\lambda_m(D)$ is not defd because 1_D is not int.

For example

$$D = [0,1]^2 \cap (\mathbb{Q} \times \mathbb{Q})^c \subset \mathbb{R}^2$$

1_D is the Dirichlet funct

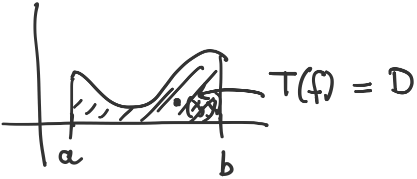
and we "know" that 1_D is not int $\Rightarrow \nexists\ \lambda_2(D)$.

Rmk: The interpretation of $\lambda_m(D)$ is

- for $m=1$, $\lambda_1(D)$ is an extension of length of D
- for $m=2$, $\lambda_2(D)$ " " " of area
- " $m=3$, $\lambda_3(D)$ " " " " volume

Case $m=2$. In first year Calc., if $f:[a,b]\to[0,+\infty[$, integrable on $[a,b]$

$$\int_a^b f(x)\,dx = \text{Area}\,(T(f))$$



$$D = T(f) = \left\{ (x,y) : \underline{a \leq x \leq b,\ \boxed{0 \leq y \leq f(x)}} \right\}$$

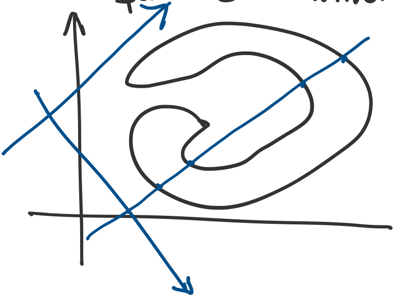
$$\boxed{\lambda_2(D)} = \int_D 1 \, dx\,dy$$

$$\overset{RF}{=} \int_a^b \underbrace{\int_0^{f(x)} 1 \, dy}_{\overset{\|}{f(x)}} \, dx$$

$$= \int_a^b f(x)\,dx. = \text{Area}(T(f))$$

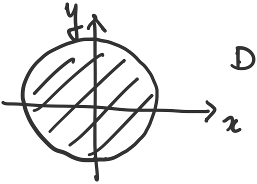
$\lambda_2(D)$ coincides with old def of area when this last is defd.

However, $\lambda_2(D)$ exists also for D which are not $T(f)$



Examples:

- Area of a disk of radius r $= \pi r^2$



$$D = \{(x,y) : \underline{x^2+y^2 \leq r^2}\}$$

$$\lambda_2(D) = \int_{\underline{\underline{D}}} \textcircled{1}\, dx\, dy$$

$$= \int \quad \int 1 \; dy \quad dx$$

$$x^2+y^2 \leq r^2 \iff y^2 \leq r^2 - x^2$$

$$\iff -\sqrt{r^2-x^2} \leq y \leq \sqrt{r^2-x^2}$$

$$r^2 - x^2 \geq 0 \iff x^2 \leq r^2$$

$$\iff |x| \leq r$$

$$x^2+y^2 \leq r^2 \iff \begin{matrix} -\sqrt{r^2-x^2} \leq y \leq \sqrt{r^2-x^2} \\ -r \leq x \leq r \end{matrix}$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} 1 \; dy \quad dx$$

$$= \int_{-r}^{r} 2 \underset{\curvearrowleft}{\sqrt{r^2-x^2}} \; dx$$

$$= 2r \int_{-r}^{r} \sqrt{1-\left(\frac{x}{r}\right)^2}\ dx$$

$$= 2r \int_{-1}^{1} \sqrt{1-u^2}\ r\, du$$

$u = x/r$

$dx = r\,du$

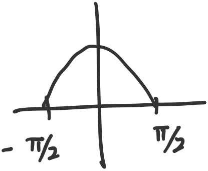
$$= 2r^2 \int_{-1}^{1} \sqrt{1-u^2}\ du$$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \underset{\substack{\| \\ \sqrt{\cos^2\theta} \\ \| \\ |\cos\theta|}}{\sqrt{1-\sin^2\theta}}\ \cos\theta\ d\theta$$

$\theta = \arcsin u \quad u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \cancel{|\cos\theta|}\cos^2\theta\ d\theta = \pi/2$$



$$\int_{-\pi/2}^{\pi/2} \cos^2\theta\ d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta(\sin\theta)'d\theta$$

$$= \underset{\substack{\| \\ 0}}{\cos\theta\sin\theta}\Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} -\sin^2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sin^2 \theta \, d\theta$$

$$1 - \cos^2 \theta$$

$$= \pi - \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta = \pi/2$$

$$\Rightarrow \quad \lambda_2 (D) = \pi r^2 . \quad \blacksquare$$

Volume of a sphere of radius r : $\frac{4}{3}\pi r^3$

$$\lambda_3 (D) = \int_D 1 \, dx\,dy\,dz$$

$$D = \{ (x,y,z) \in \mathbb{R}^3 : \underbrace{x^2+y^2+z^2 \leq r^2}_{x^2+y^2 \leq r^2 - z^2} \}$$

$$\overset{RF}{=} \int_{\substack{r^2 - z^2 \geq 0 \\ z^2 \leq r^2}} \int_{x^2+y^2 \leq r^2 - z^2} 1 \quad dx\,dy \quad dz$$

$$= \int_{-r}^{r} \left(\int_{x^2+y^2 \leq r^2 - z^2} 1 \, dx\, dy \right) dz$$

$$\Big(\quad \lambda_2\left(\{x^2+y^2 \leq r^2 - z^2\}\right)$$

$$= \pi \cdot (r^2 - z^2)$$

$$= \int_{-r}^{r} \underline{\pi(r}^2 - z^2)\, dz$$

$$= \pi r^2 \cdot \underbrace{\int_{-r}^{r} 1 \, dz}_{2r} - \pi \int_{-r}^{r} z^2 \, dz$$

$$= 2\pi r^3 - \frac{\pi}{3} \, \frac{z^3}{\cancel{3}} \Big|_{z=-r}^{z=+r}$$

$$(r^3 - (-r^3))$$

$$2r^3$$

$$= 2\pi r^3 - \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3. \qquad \blacksquare$$

Do Ex 5.7.2 #2,3.