COMPUTABILITY (19/12/2023)

A typical exam...

# Computability
## Jan 19 2022

definitions
proofs
small variations

**Exercise 1**

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if $A$ is not recursive and $A \leqslant_m B$ then $B$ is not recursive.

c. Show that if $A$ is recursive then $A \leqslant_m \{1\}$.

constructions of PR / R
diagonalisation
smn

**Exercise 2**

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs $x$, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

classify sets (recursive r.e.), saturatedness

**Exercise 3**

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether $A$ and $\bar{A}$ are recursive/recursively enumerable.

**Exercise 4**

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether $B$ and $\bar{B}$ are recursive/recursively enumerable.

*Note: Each exercise contributes with the same number of points (8) to the final grade.*

$\boxed{\text{ORAL EXAM}}$: optional, needed for distinction (lode)
focused on theory/proofs
range: +/- 4

## Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if $A$ is not recursive and $A \leqslant_m B$ then $B$ is not recursive.

c. Show that if $A$ is recursive then $A \leqslant_m \{1\}$.

(a) We say

$$A \leq_m B$$

if there exists a total computable function $f: \mathbb{N} \to \mathbb{N}$

s.t. for all $x \in \mathbb{N}$

$$x \in A \quad \text{if and only if} \quad f(x) \in B$$

(b) We prove the counternominal, i.e. if $A \leq_m B$ and $B$ recursive then $A$ is recursive

Assume $A \leq_m B$ and let $f: \mathbb{N} \to \mathbb{N}$ be the reduction function i.e.

$$\forall x \in \mathbb{N} \qquad x \in A \quad \text{iff} \quad f(x) \in B \qquad (*)$$

assume $B$ recursive, i.e.

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases} \quad \text{is computable}$$

observe that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \overset{(*)}{=} \begin{cases} 1 & \text{if } f(x) \in B \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_B(f(x))$$

since $\chi_A$ is the composition of computable functions, it is computable

$\Rightarrow$ $A$ is recursive.

(c) $A$ is recursive $\Rightarrow$ $A \leq_m \{1\}$

if $A$ is recursive, then

$\chi_A : \mathbb{N} \to \mathbb{N}$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

is computable and total

$x \in A$ iff $\chi_A(x) = 1$ iff $\chi_A(x) \in \{1\}$

hence $\chi_A$ is the reduction function for $A \leq_m \{1\}$

Extra question: Does the converse hold?

$A \leq_m \{1\}$ then $A$ is recursive

yes, since $\{1\}$ is finite, hence it is recursive.

(or, alternatively: let $f : \mathbb{N} \to \mathbb{N}$ be the reduction function for $A \leq_m \{1\}$

Then $\forall x$

$x \in A$ iff $f(x) \in \{1\}$ iff $f(x) = 1$

thus

$$\chi_A(x) = \overline{sg}(|f(x) - 1|) = \begin{cases} 1 & \text{if } f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

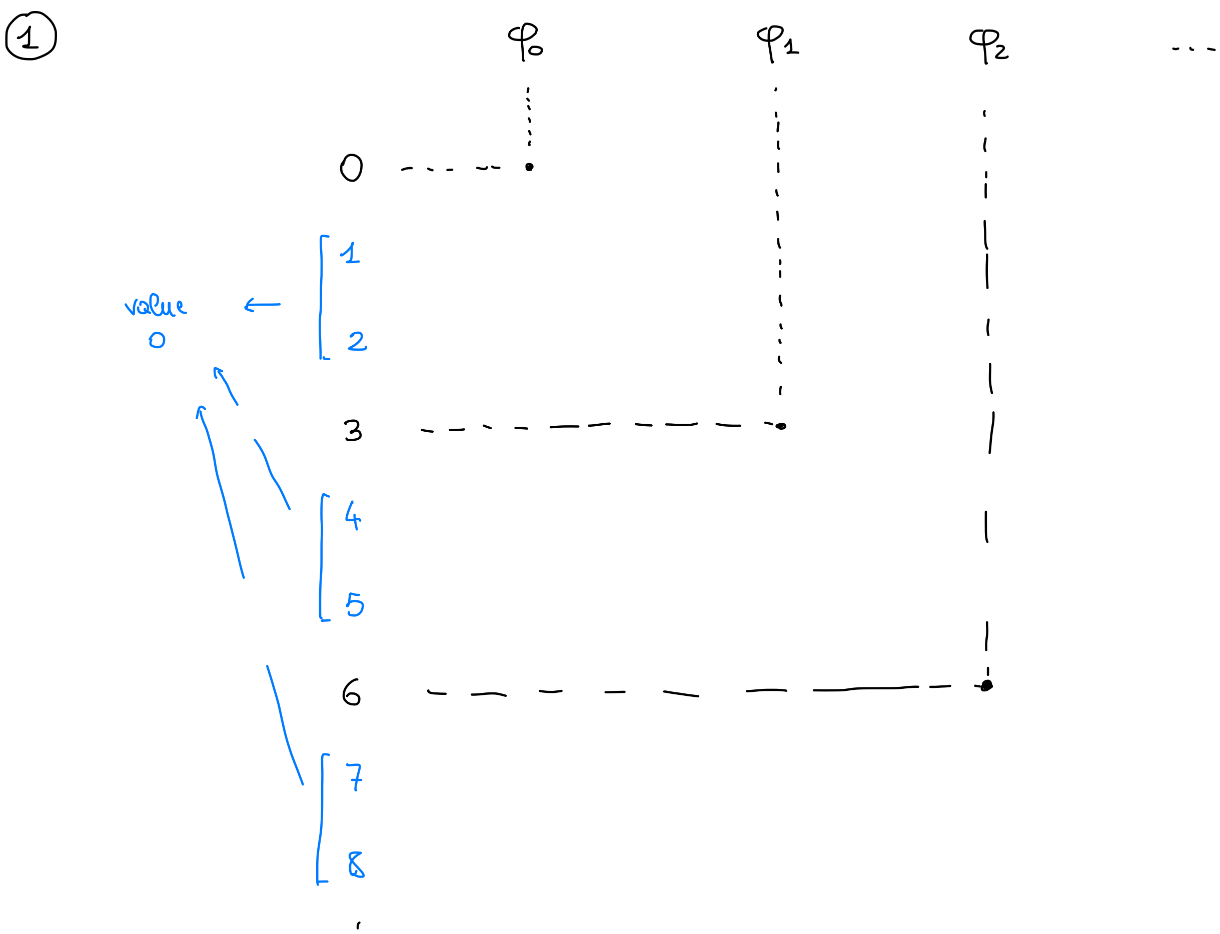
$$= \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

## Exercise 2

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs $x$, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

Idea :

1




$$g(x) = \begin{cases} \varphi_{y = x/3}(x) + 1 & \text{if } x = 3y \text{ for some } y \text{ and } \varphi_y(x)\downarrow \\ 0 & \text{if } (x = 3y \text{ for some } y \text{ and } \varphi_y(x)\uparrow) \text{ or } x \neq 3y \ \forall y \end{cases}$$

note that $g$ is

→ total

→ not computable since $\forall y \quad \varphi_y(3y) \neq g(3y)$

- if $\varphi_y(3y)\downarrow$ then $g(3y) = \varphi_y(3y) + 1$
- if $\varphi_y(3y)\uparrow$ then $g(3y) = 0$

→ there are infinitely many $x$ s.t. $g(x) = g(x+1)$

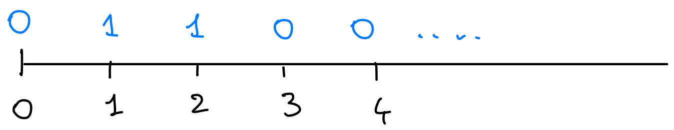
$\forall y$ if $x = 3y+1$

neither $x$ nor $x+1$ are multiples of 3

hence $g(x) = g(x+1) = 0$

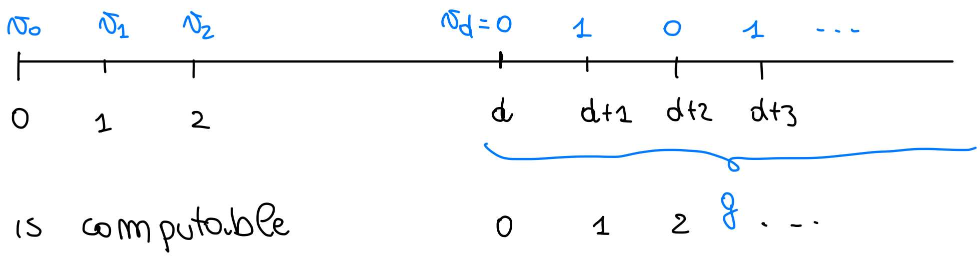
## (2) alternative solution

Can I use $\chi_K$ ?




OBSERVATION : let $f : \mathbb{N} \to \mathbb{N}$ be a function s.t. $cod(f) = \{0,1\}$

and there is $d \in \mathbb{N}$ s.t. $\forall x > d$ $f(x) \neq f(x+1)$




Then $f$ is computable

In fact, let

$$f(x) = v_x \qquad x \leq d \qquad \text{and} \qquad v_d = 0$$

and define $g : \mathbb{N} \to \mathbb{N}$

$$\begin{cases} g(0) = 0 \\ g(y+1) = \overline{sg}(g(y)) \end{cases} \qquad \text{computable}$$

Then

$$f(x) = \prod_{i=0}^{d-1} \overline{sg}(|x-i|) \cdot v_i \; + \; g(x \dot{-} d)$$

computable !

Hence the desired function in the exercise can be $f = \chi_K$

- $\chi_K$ total
- $\chi_K$ non computable
- $\forall d \quad \exists x \geq d \quad$ s.t. $\chi_K(x) = \chi_K(x+1)$

  (otherwise it would be computable)

  $\Rightarrow \quad \{x \in \mathbb{N} \mid \chi_K(x) = \chi_K(x+1)\}$ is infinite

**Exercise 3**

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether $A$ and $\bar{A}$ are recursive/recursively enumerable.

conjecture : $A$ not r.e.

$\bar{A}$ not r.e.

$A$ is saturated

$A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

$\mathcal{A} = \{ f \mid f \text{ is quasi total} \} = \{ f \mid \overline{dom(f)} \text{ finite} \}$

* $A$ is not r.e.

observe that $id \in \mathcal{A}$ since $\overline{dom(id)} = \overline{\mathbb{N}} = \emptyset$ is finite

and for all $\vartheta \subseteq id$, $\vartheta$ finite $\vartheta \notin \mathcal{A}$ since

$dom(\vartheta)$ is finite

$\Downarrow$

$\overline{dom(\vartheta)}$ infinite

hence $A$ is not r.e., by Rice-Shapiro

• $\bar{A}$ is not r.e. ( $\bar{\mathcal{A}} = \{ f \mid f \text{ not quasi-total} \}$

$= \{ f \mid \overline{dom(f)} \text{ infinite} \}$ )

note that $id \notin \bar{\mathcal{A}}$

and $\vartheta = \emptyset \subseteq id$ finite and $\vartheta \in \bar{\mathcal{A}}$

since $\overline{dom(\vartheta)} = \bar{\emptyset} = \mathbb{N}$ infinite

hence, by Rice-Shapiro, $\bar{A}$ is not r.e.

**Exercise 4**

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether $B$ and $\bar{B}$ are recursive/recursively enumerable.

conjecture : $\boxed{B \text{ is r.e. , not recursive}}$

$\Downarrow$

$\bar{B}$ not r.e. (otherwise $B$ recursive)

and thus $\bar{B}$ not recursive

* <u>B is r.e.</u>

$y = 2x+1+d$

in fact

$$sc_B(x) = \mathbb{1}\left(\mu(z,y,t).\ (S(x,z,y,t) \wedge y > 2x)\right)$$

$$= \mathbb{1}\left(\mu(z,d,t).\ S(x,z,2x+1+d,t)\right)$$

$$= \mathbb{1}\left(\mu\omega.\ S(x,(\omega)_1, 2x+1+(\omega)_2, (\omega)_3)\right)$$

$$= \mathbb{1}\left(\mu\omega.\ |\chi_S(x,(\omega)_1, 2x+1+(\omega)_2, (\omega)_3) - 1|\right)$$

computable , hence $B$ is r.e.

* <u>B is not recursive</u>

We show that $K \leq_m B$

we need a total computable function $s: \mathbb{N} \to \mathbb{N}$ s.t.

$$x \in K \quad \text{iff} \quad s(x) \in B$$

$$\exists z.\ \varphi_{s(x)}(z) > 2\,s(x)$$

define

$$g(x,z) = \begin{cases} z & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

$$= z \cdot sc_K(x)$$

hence $g$ is computable

By the smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ total and computable s.t. $\forall x, z$

$$\varphi_{s(x)}(z) = g(x,z) = \begin{cases} z & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

We claim that $s$ is the reduction function $K \leq_m B$

* if $x \in K$ then $s(x) \in B$

  let $x \in K$. Then $\varphi_{s(x)}(z) = z \quad \forall z$

  hence $\varphi_{s(x)}(2s(x)+1) = 2s(x)+1 > 2s(x)$

  thus $s(x) \in B$

* if $x \notin K$ then $s(x) \notin B$

  let $x \notin K$. Hence

  $\varphi_{s(x)}(z) \uparrow \quad \forall z$

  Thus we have $E_{s(x)} = \emptyset$, hence there is no $y \in E_{s(x)}$ s.t. $y > 2s(x)$

  hence $s(x) \notin B$

Hence $B$ is not recursive.

Since $B$ is r.e. and not recursive, then $\overline{B}$ not r.e.

(otherwise if $B, \overline{B}$ r.e. we would have $B$ recursive)

In turn, this implies that $\overline{B}$ not recursive.

* EXTRA QUESTION: Is $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ saturated ?

Apparently it is not since it "refers to $x$ in the property"

Let's prove it by showing that there are

$$e \in B \qquad e' \notin B \qquad \text{with } \varphi_e = \varphi_{e'}$$

We show that there is $e \in \mathbb{N}$ s.t.

$$\varphi_e(x) = 2e + 1$$

Define

$$g(m, x) = 2m + 1$$

computable, hence by smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total and computable s.t. $\forall m, x$

$$\varphi_{s(m)}(x) = g(m, x) = 2m + 1$$

Since $s$ is total and computable, there is $e \in \mathbb{N}$ st.

$$\varphi_{s(e)} = \varphi_e$$

Thus

$$\varphi_e(x) = \varphi_{s(e)}(x) = 2e + 1$$

Hence

$$e \in B \qquad \text{since} \quad \underset{2e}{\underset{\vee}{2e+1}} \in E_e$$

Now, there are infinitely many indexes for $\varphi_e$, thus we can take $e' \in \mathbb{N}$ $e' > e$ s.t. $\varphi_e = \varphi_{e'}$

Note that $E_{e'} = E_e = \{2e+1\}$ and $2e+1 < 2e'$

thus $e' \notin B$

Summing up $e, e' \in \mathbb{N}$ $e \in B$, $e' \notin B$ $\varphi_e = \varphi_{e'}$

$\Rightarrow$ B is not saturated