Monday, 18 December 2023 08:40

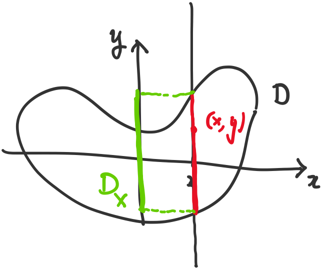
Reduction Formula

f integrable on D $\left(\exists \int_D f\right)$, $f = f(x,y)$

then

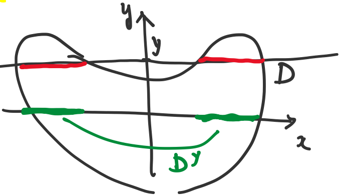
$$\int_D f = \int_{x:\, D_x \neq \emptyset} \int_{D_x} f(x,y)\,dy\,dx$$

$D_x = \{y : (x,y) \in D\}$



$$= \int_{y:\, D^y \neq \emptyset} \int_{D^y} f(x,y)\,dx\,dy \,.$$

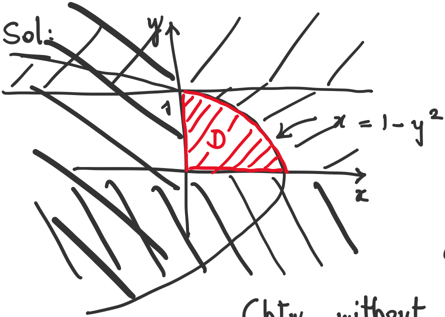
$D^y = \{x : (x,y) \in D$



Ex 5.7.1

$$1.\quad \int_{\underbrace{0\le y\le 1,\ 0\le x\le 1-y^2}} x e^{y}\, dx dy .$$

Sol:



$x = 1 - y^2$

D is clearly closed & bded

(btw, without any figure,

D is closed because it is defd by large ineq. involving cont functs,

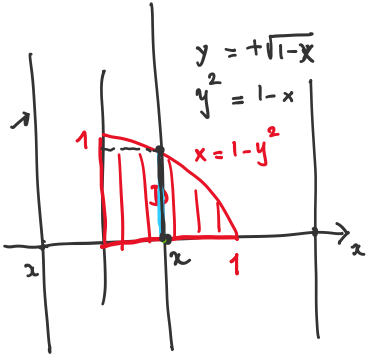
D is bded, because

$$(x,y)\in D \Rightarrow \begin{array}{l} 0\le y\le 1 \quad (y \text{ bded}) \\ 0\le x\le 1-y^2\le 1 \\ \qquad (x \text{ bded})) \end{array}$$

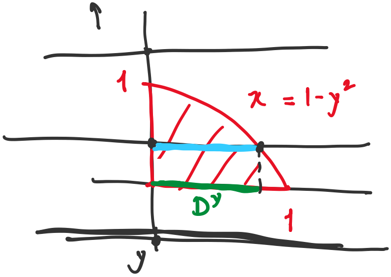
$f(x,y) = x e^{y} \in \mathcal{C}(D) \Rightarrow$ f is int. on D

$\exists \int_D f$.

To compute the int we apply RF:



$$D_x = \begin{cases} \emptyset & x<0 \\ 0\le y\le\sqrt{1-x} & 0\le x\le 1 \\ \emptyset & x>1 \end{cases}$$



$$D^y = \begin{cases} \emptyset & y<0 \\ 0\le x\le 1-y^2 & 0\le y\le 1 \\ \emptyset & y>1 \end{cases}$$

$$\int_D f = \int_{x:D_x\neq\emptyset} \left(\int_{D_x} x e^y \, dy \right) dx$$

$$= \int_0^1 \left(\int_0^{\sqrt{1-x}} x e^y \, dy \right) dx$$

$$= \int_0^1 x \left[e^y \right]_{y=0}^{y=\sqrt{1-x}} dx$$

$$\left(e^{\sqrt{1-x}} - e^0 \right)$$

$$= \int_0^1 x \left(e^{\sqrt{1-x}} - 1 \right) dx.$$

$$\int_D f = \int_{y : D^y \neq \phi} \left(\int_{D^y} x e^y \, dx \right) dy$$

$$\boxed{= \int_0^1 \int_0^{1-y^2} x e^y \, dx \quad dy}$$

$$= \int_0^1 e^y \int_0^{1-y^2} x \, dx \quad dy$$

$$\|$$

$$\frac{x^2}{2} \Big|_0^{1-y^2} = \frac{1}{2}\left((1-y^2)^2 - 0 \right)$$

$$= \frac{1}{2} \int_0^1 (1-y^2)^2 e^y \, dy \overset{(*)}{=}$$

$$\|$$

$$\left(\underset{\sim}{1} + y^4 - 2y^2 \right) \underset{\smile}{e}^y$$

$$\int_0^1 y^4 \underset{\partial_y(e^y)}{\underset{\shortparallel}{e^y}}\, dy = y^4 e^y \Big|_0^1 - \int_0^1 4y^3 \underbrace{e^y}\, dy$$

$$= e - 4 \int_0^1 y^3\, \partial_y e^y\, dy$$

$$= e - 4\left[\, y^3 e^y \Big|_0^1 - \int_0^1 3y^2 e^y dy \right]$$

$$= e - 4\left[e - 3 \int_0^1 y^2 e^y\, dy\right]$$

$$= -3e + 12 \int_0^1 y^2 e^y\, dy$$

$$\int_0^1 y^2 \underset{\partial_y e^y}{\underset{\shortparallel}{e^y}}\, dy = y^2 e^y \Big|_0^1 - \int_0^1 2y e^y\, dy$$

$$= e - 2 \int_0^1 y e^y\, dy$$

$$= e - 2\left[\, \underset{e}{\underset{\shortparallel}{y e^y \Big|_0^1}} - \int_0^1 1 \cdot e^y\, dy \right]$$

$$= -e + 2 \int_0^1 e^y\, dy = -e + 2(e-1)$$

$$\int_0^1 e^y\, dy = e^y \Big|_0^1 = \underline{e-1} \qquad = e - 2$$

$$\int_0^1 y^4 e^y\,dy = -3e + 12(e-2) = 9e - 24$$

$$\overset{(*)}{=} \frac{1}{2}\Big[\, e-1 + 9e-24 - 2(e-2)\Big].$$

■

Rmk: We can apply RF without drawing D:

$$\int_D f = \int_{\boxed{0\leqslant y\leqslant 1}\,\boxed{0\leqslant x\leqslant 1-y^2}} xe^y\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y^2} xe^y\,dx\,dy$$

correct

$$= \int_0^{1-y^2} \left(\int_0^1 xe^y\,dy\right) dx$$

wrong!

$$\int_0^{1-y^2} x \int_0^1 e^y\,dx = (e-1)\,\frac{x^2}{2}\Big|_0^{1-y^2}$$

$$\text{II}$$

$(e-1)$

$$= \frac{e-1}{2}(1-y^2)$$

!!!

∎

2. $$\int_{\boxed{0\leq y \leq 1-x^2}} \underline{\frac{x}{2+y}}\, dxdy$$

Sol: To apply RF we need to know integrability of f.

$$D = \{(x,y) : 0 \leq y \leq 1-x^2\}$$

D is closed because it is defd by large ineq.

D is bded because if $(x,y) \in D$

$$\Rightarrow \quad 0 \leq y \leq 1-x^2 \leq 1 \Rightarrow 0 \leq y \leq 1$$

$$1-x^2 \geq \cancel{y} \geq 0 \Rightarrow x^2 \leq 1$$

$$\Rightarrow -1 \leq x \leq 1$$

$$f \in \mathcal{C}(\mathbb{R}^2 \setminus \{y=-2\}) \Rightarrow f \in \mathcal{C}(D)$$

$\Rightarrow f$ is int on D.

We apply now RF to compute the int

$$\int_D f = \int_{0 \leq y \leq 1-x^2,\ -1 \leq x \leq 1} \frac{x}{2+y}\, dx\, dy$$

$$= \int_{-1}^{1} \left(\int_0^{1-x^2} \frac{\boxed{x}}{2+y}\, dy \right) dx$$

$$= \int_{-1}^{1} x \int_0^{1-x^2} \underset{\substack{\| \\ \partial_y \log|2+y|}}{\frac{1}{2+y}}\, dy \quad dx$$

$$= \int_{-1}^{1} x \quad \log|2+y| \Big|_{y=0}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} x \left(\log|\underset{\geq 0}{3-x^2}| - \log 2 \right) dx$$

$$= \underset{\substack{\| \\ 0}}{\int_{-1}^{1} x \log(3-x^2)} - \log 2 \underset{\substack{\| \\ 0}}{\int_{-1}^{1} x\, dx}$$

$$\Gamma \quad \int^a g(t)\, dt = 0 \qquad g \text{ odd}$$

$$= 0$$

■

$-a$

$g(-t) = -g(t)$

$$\int_{-a}^{a} g(t)\,dt = 2\int_{0}^{a} g(t)\,dt \qquad g \text{ even}$$

$g(-t) = g(t)$ ⌟

#3 $$\int_{|y| \leq 1-x^2} \frac{1}{1+y}\,dx\,dy$$

Sol: Integrability : $D = \{(x,y) : \ |y| \leq 1-x^2\}$

D is closed (defd by large ineq with cont functs)

D is bded : $0 \leq |y| \leq 1 - x^2 \leq 1$

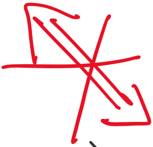
$\Rightarrow |y| \leq 1$

$1 - x^2 \geq 0 \Rightarrow |x| \leq 1$

Rmk: A frequent error is

$|y| \leq 1 - x^2 \Rightarrow |y| \leq 1$

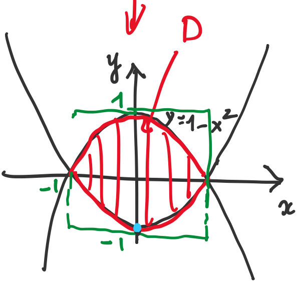
$-(1-x^2) \leq y \leq 1-x^2$

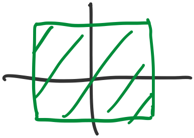


$1 - x^2 \geq 0 \Rightarrow x^2 \leq 1$

$\Rightarrow |x| \leq 1$

$\Rightarrow -1 \leq y \leq 1, \ -1 \leq x \leq 1$





$$f(x,y) = \frac{1}{1+y} \in \mathcal{C}(\mathbb{R}^2 \setminus \{y=-1\})$$

$\Rightarrow f \notin \mathcal{C}(D).$ How can we do in this case to discuss int ?

$\ulcorner$ f int $\Leftrightarrow \exists \int_D |f| < +\infty$

If $\int_D |f| < +\infty$

$$\|$$

$$+\infty > \int_D |f(x,y)|\, dx\, dy \overset{RF}{=}$$

RF applies to $|f|$

$$= \int_{x:D_x \neq \phi} \left(\int_{D_x} |f(x,y)| \, dy \right) dx$$

$$= \int_{y:D^y \neq \phi} \left(\int_{D^y} |f(x,y)| \, dx \right) dy$$

So

$$\int_D |f| < +\infty \implies \int_{x:D_x \neq \phi} \int_{D_x} |f| \, dy \, dx < +\infty$$

$$\int_{y:D^y \neq \phi} \left(\int_{D^y} |f| \, dx \right) dy < +\infty$$

Thm: If one of the two iterated integrals

$$\int_{x:D_x \neq \phi} \int_{D_x} |f| \, dy \, dx \quad , \quad \int_{y:D^y \neq \phi} \int_{D^y} |f| \, dx \, dy$$

is finite. Then f is int. on D

$\left(\int_D |f| < +\infty \right)$.

Rmk: In part, if f has constant sign, for example $f \geq 0$, $|f| = f$.

In this case, to check int. it is suff to check that one of the ints

$$\int_{D_x \neq \phi} \Big(\int_{D_x} f \, dy \Big) \, dx , \qquad \int_{D^y \neq \phi} \Big(\int_{D^y} f \, dx \Big) \, dy$$

is finite to have, at once

- integrability
- the value of $\int_D f \overset{RF}{=}$

But CAREFUL ! If f has var. sign it might happens that

$$\int_{D_x \neq \phi} \int_{D_x} f \, dy \, dx \quad \text{and} \quad \int_{D^y \neq \phi} \int_{D^y} f \, dx \, dy$$

are both FINITE but f is NOT integrable (example below).

⌟

Let's go back to

$$\int_{|y|\leqslant 1-x^2} \frac{1}{1+y}\,dx\,dy$$

$f(x,y) = \frac{1}{1+y}$ on $D = \{(x,y) : |y| \leqslant 1-x^2\}$

$f \geqslant 0$

$(f \geqslant 0 \Leftrightarrow 1+y>0 \Leftrightarrow y>-1)$

To check int we compute

$$\boxed{\int_{D_x \neq \emptyset} \int_{D_x} \underset{f}{\underset{\parallel}{|f|}}\,dy\,dx}$$ or $$\int_{D^y \neq \emptyset} \left(\int_{D^y} \underset{f}{\underset{\parallel}{|f|}}\,dx\right)dy$$

$D = \{(x,y) : |y| \leqslant 1-x^2\}$

$\{-(1-x^2) \leqslant y \leqslant 1-x^2, \quad -1 \leqslant x \leqslant 1\}$

$$\int_{-1}^{1}\left(\int_{-(1-x^2)}^{1-x^2} \underset{\partial_y \log|1+y|}{\underset{\parallel}{\frac{1}{1+y}}}\,dy\right)dx$$

$$= \int_{-1} \log|1+y| \Big|_{y=-(1-x^2)=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \log(2-x^2) - \log x^{\boxed{2}} \, dx$$

$$= 2\int_0^1 1\cdot\log(2-x^2) - \underbrace{1\cdot 2}\log x \, dx$$

$$= 2\Big[\, x\overset{0}{\overset{\parallel}{\log}}(2-x^2)\Big|_0^1 - \int_0^1 x\cdot\frac{-2x}{2-x^2}\,dx$$

$$-2\left(x\log x\Big|_0^1 - \int_0^1 \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right)\Big]$$

$$= 2\Big[\, 0-0 + 2\int_0^1 \frac{x^2}{2-x^2}\,dx$$

$$\underbrace{-2(0-0-1)}_{+2}\Big]$$

$$= 4\left(\int_0^1 \frac{x^2}{2-x^2}\,dx + 1\right)$$

$$-\int_0^1 \frac{2-x^2-2}{2-x^2}\,dx = -\int_0^1 1\,dx - 2\int_0^1 \frac{1}{(x-\sqrt{2})(x+\text{[illegible]}}$$

$$= -1 - \frac{2}{2\sqrt{2}}\int_0^1 \frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}$$

$$= -1 - \frac{1}{\sqrt{2}}\left(\log|x-\sqrt{2}|\Big|_0^1 - \log|x+\sqrt{2}|\Big|_0^1\right)$$

$$= -1 - \frac{1}{\sqrt{2}}\left(\log(\sqrt{2}-1) - \log\sqrt{2} - \left(\log(\sqrt{2}+1) - \log\sqrt{2}\right)\right)$$

$$= -1 - \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$\Rightarrow \int_{D_x \neq \phi}\int_{D_x} \underset{\substack{\| \\ f}}{|f|}\,dy\,dx = \qquad f \geq 0 \text{ on } D$$

$$= 4\left[\cancel{-1} + \frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\ \cancel{+1}\right]$$

$$= \frac{4}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\,. < +\infty$$

$\Rightarrow$ f is int on D and

$$\int_D f \overset{RF}{=} \int_{D_x \neq \emptyset} \int_{D_x} f \, dy \, dx = \frac{4}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$\blacksquare$

Do Ex 5.7.1 # 4, 5, 6, 7, 8, 9

↑

tricky

Rmk: If f has not const sign we might have

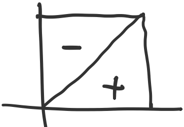
$$\int \left(\int f \, dy \right) dx \quad \text{and} \quad \int \left(\int f \, dx \right) dy$$

are finite but f is not int!

Ex 5.3.7

$$f(x,y) = \frac{x-y}{(x+y)^3} \qquad (x,y) \in D = [0,1]^2$$

You can verify that



$$\int_{x : D_x \neq \emptyset} \left(\int_{D_x} f(x,y) \, dy \right) dx = 1/2$$

$$\neq$$

$$\int_{y : D^y \neq \emptyset} \left(\int_{D^y} f(x,y) \, dx \right) dy = -1/2$$

From this it follows that f cannot be int otherwise

$$\int_D f \overset{RF}{=} \int \int f \, dy \, dx = \tfrac{1}{2} \quad !$$

$$= \int \int f \, dx \, dy = -\tfrac{1}{2}$$

$\blacksquare$