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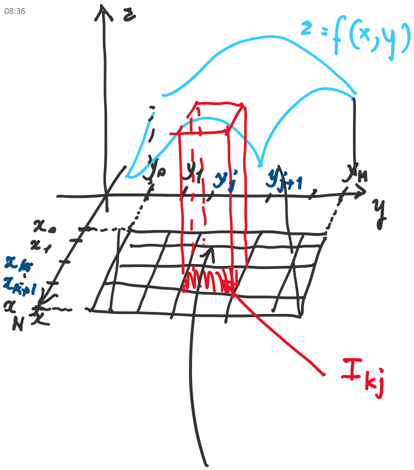


$$\mathrm{Vol}_{k,j} = \underbrace{(x_{k+1} - x_k)}_{dx_k}\underbrace{(y_{j+1} - y_j)}_{dy_j} h_{k,j}$$

$$h_{k,j} = \inf_{(x,y)\in I_{kj}} f(x,y)$$

$$\sum_{k,j} \mathrm{Vol}_{k,j} =: \underline{S}(\pi)$$

$$\boxed{\underline{S}(\pi)} = \sum_{k,j} \boxed{h_{k,j}}\, dx_k\, dy_j .$$

$$\underline{A}(T(f)) = \operatorname{Sup}\left\{ \underline{S}(\pi) \ :\ \pi \text{ partition of } I \right\}$$

(the best approx from below of $A(T(f))$)

Similarly we define

$$\boxed{\overline{S}(\pi)} = \sum_{k,j} H_{k,j} \, dx_k \, dy_j$$

$$H_{k,j} = \sup_{(x,y) \in I_{k,j}} f(x,y)$$

$$\overline{A}(T(f)) := \text{Inf} \left\{ \overline{S}(\pi) \; : \; \pi \text{ partitions of } I \right\}$$

(the best approx from above of $A(T(f))$).

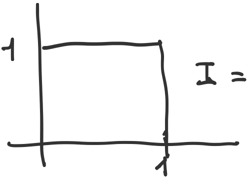
Prop:

$$\underline{A}(T(f)) \leq \overline{A}(T(f))$$

Rmk: $\underline{A}(T(f)) < \overline{A}(T(f))$ may happens

For ex.

$$f(x,y) = \begin{cases} 1 & (x,y) \in (\mathbb{Q} \times \mathbb{Q})^c \cap I \\ 0 & (x,y) \in (\mathbb{Q} \times \mathbb{Q}) \cap I \end{cases}$$



$I = [0,1]^2$

In this case

$$\underline{A}(T(f)) = 0 \qquad \bar{A}(T(f)) = 1.$$

■

Def: If $\underline{A}(T(f)) = \bar{A}(T(f))$, we say that $A(T(f))$ is well defd and

$$A(T(f)) = \underline{A}(T(f)) = \bar{A}(T(f)).$$

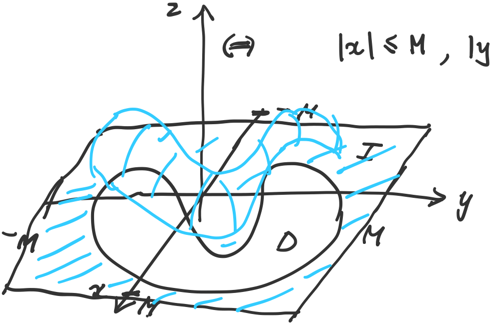
Thm: If $f \in \mathcal{C}(I)$ (I int in $\mathbb{R}^2$), $f \geqslant 0 \Rightarrow \exists\, A(T(f))$.

The next step is to extend this def to the case when $f = f(x,y) : D \subset \mathbb{R}^2 \to [0,+\infty[$

Case 1: D bounded

$$\Leftrightarrow \quad \|(x,y)\| \leqslant M \qquad \forall (x,y) \in D$$

$\Leftrightarrow \quad |x| \leq M, \ |y| \leq M \quad \forall (x,y) \in D$



Define

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & (x,y) \in D \\ 0 & (x,y) \in I \setminus D \end{cases}$$

$$= f(x,y) \cdot \begin{cases} 1 & (x,y) \in D \\ 0 & (x,y) \in I \setminus D \end{cases}$$

$$1_D(x,y) := \begin{cases} 1 & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

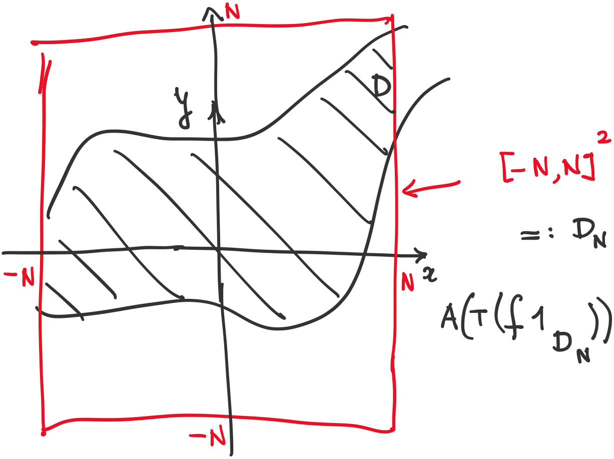
$$\tilde{f} = f \cdot 1_D$$

Def: If $f : D \subset \mathbb{R}^2 \longrightarrow [0, +\infty[$, D bounded, we say that $\exists A(T(f))$ if

$$\exists \quad A(T(\underline{f \cdot 1_D}))$$

↑ this is defd on I interval

Case 2 : D unbded



$[-N,N]^2 \cap D$

$=: D_N$

$A(T(f 1_{D_N}))$

Def: $f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow [0,+\infty[$, D unbded

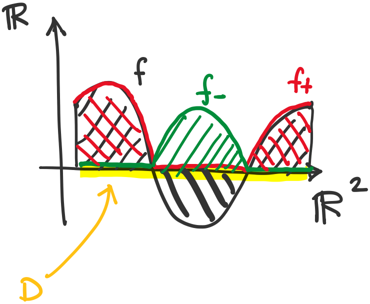
$$A(T(f)) = \lim_{N \to +\infty} A(T(f 1_{D_N}))$$

$D_N = D \cap [-N,N]^2$.

Let's say that $A(T(f))$ is now defd for $f : D \subset \mathbb{R}^2 \longrightarrow [0,+\infty[$.

Integral

Let's now consider $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$.



Def:

$$f_+ = \begin{cases} f & \text{if } f \geq 0 \\ 0 & \text{if } f < 0 \end{cases} \quad (\text{pos. part of } f)$$

$$f_- = \begin{cases} 0 & \text{if } f \geq 0 \\ -f & \text{if } f < 0 \end{cases} \quad (\text{neg part of } f$$

We notice that

- $f_+ , f_- \geq 0$
- $\boxed{A(T(f_+))}$ is the vol of red part
- $\boxed{A(T(f_-))}$ " " " " green part

Def: We say that $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is (Riemann) integrable on D if

- f_+, f_- have $A(T(f_+))$, $A(T(f_-))$
- $A(T(f_+))$, $A(T(f_-)) < +\infty$

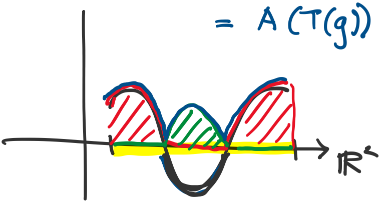
We set

$$\int_D f := A(T(f_+)) - A(T(f_-)) \in \mathbb{R}$$

Rmk:

$$A(T(f_+)), A(T(f_-)) < +\infty \iff$$

$$\underbrace{A(T(f_+)) + A(T(f_-))}_{= A(T(g)) = A(T(|f|))} < +\infty$$



$$g = \begin{cases} f & f \geq 0 \\ -f & f < 0 \end{cases} = |f|$$

In other words, we say that f is integrable on D if

$$A(T(|f|)) < +\infty$$

we set

$$\int_D f := A(T(f_+)) - A(T(f_-)).$$

Rmk: If $f \geqslant 0$ $\int_D f \underset{\substack{f_+=f \\ f_-=0}}{=} A(T(f))$

In part: $A(T(|f|)) < +\infty \iff \int_D |f| < +\infty$

Prop: (fund properties of int)

i) (linearity) f, g are int on D

$\Rightarrow$ $\alpha f + \beta g$ is int on $\forall \alpha, \beta \in \mathbb{R}$

and

$$\int_D (\alpha f + \beta g) = \alpha \int_D f + \beta \int_D g$$

ii) (isotonicity) if f, g are int on D and

$f \leqslant g$ on D

$\Rightarrow$ $\int_D f \leqslant \int_D g$.

iii) (triang ineq) if f is int on $D \Rightarrow$ $|f|$ is int on D and

$$\left| \int_D f \right| \leq \int_D |f|$$

iv) (decomposition) if f is int on D_1, D_2 $D_1 \cap D_2 = \phi \Rightarrow f$ is int $D_1 \cup D_2$ and

$$\int_{D_1 \cup D_2} f = \int_{D_1} f + \int_{D_2} f.$$

How can we compute a multiple int?

I) Reduction formula

II) Change of variables.

Reduction Formula

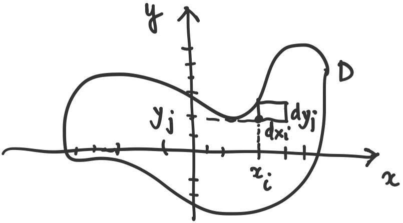
$$\int_D f$$

Prop: $f \in \mathscr{C}(D)$, D bounded $\Rightarrow$

$\forall \varepsilon > 0 \quad \exists \pi$: $\quad \left| \int_D f - \sum_{i,j} f(x_i, y_j)\, dx_i\, dy_j \right| \leq \varepsilon$

$$\left(\int_D f \approx \sum_{i,j} f(x_i, y_j)\, dx_i\, dy_j \right)$$

$$\int_D f \approx \sum_{i,j} \underset{a_{ij}}{\boxed{f(x_i, y_j)\, dx_i\, dy_j}} \leftarrow$$



$$\left.\begin{array}{ccccc}
a_{00} & a_{01} & a_{02} & - - - - & a_{0N} \\
a_{10} & a_{11} & a_{12} & & a_{1N} \\
\vdots & & & & \\
a_{M0} & a_{M1} & a_{M2} & - - - & a_{MN}
\end{array}\right.
\quad
\left.\begin{array}{l}
= \sum_j a_{0j} \\
= \sum_j a_{1j} \\
\\
= \sum_j a_{Mj}
\end{array}\right\} +$$

$$\underbrace{\sum_i a_{i0} \quad \sum_i a_{i1} \quad \sum_i a_{i2} \quad \sum a_{iN}}_{+} \qquad =$$

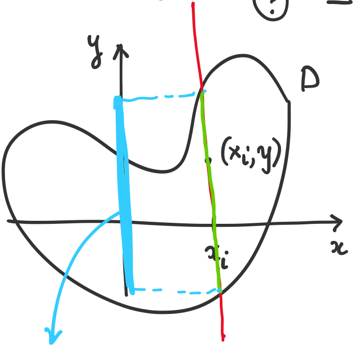
$$\sum_{i,j} a_{ij} = \sum_i \left(\sum_j a_{ij} \right) = \boxed{\sum_j \left(\sum_i a_{ij} \right)}$$

$$= \sum_i \sum_j f(x_i, y_j)\, dx_i\, dy_j$$

$$= \sum_i \left(\sum_j f(x_i, y_j)\, dy_j \right) dx_i$$

$$\sum_j f(y_j)\, dy_j \sim \int_{?} f(y)\, dy$$

$$\sum_j f(x_i, y_j)\, dy_j \sim \int_{?} f(\underline{\underline{x_i}}, y)\, dy$$



$$\{ y : (x_i, y) \in D \} =: D_{x_i} \quad (x_i\text{-sect of } D)$$

Def: $D_x = \{ y : (x,y) \in D \}$

$$\boxed{\sum_j f(x_i, y_j)\, dy_j \approx \underbrace{\int_{D_{x_i}} f(x_i, y)\, dy}_{F(x_i)}}$$

So

$$\int_D f \approx \sum_{i,j} f(x_i, y_j)\, dx_i\, dy_j$$

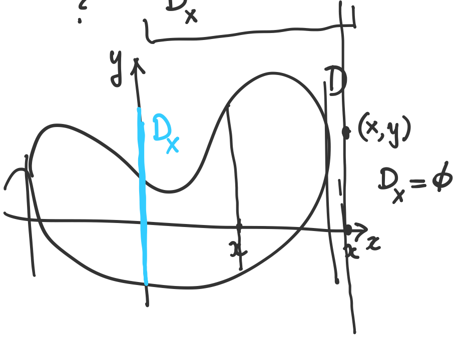
$$= \sum_i \left(\sum_j f(x_i, y_j)\, dy_j \right) \underbrace{dx_i}$$

$$\approx \int_{D_{x_i}} f(x_i, y)\, dy = F(x_i)$$

$$= \sum_i F(x_i)\, dx_i$$

$$\approx \int_{?} \underbrace{F(x)}\, dx$$

$$= \int_{?} \left(\int_{D_x} f(x,y)\,dy \right) dx$$



$$= \int_{x:\,D_x \neq \phi} \left(\int_{D_x} f(x,y)\,dy \right) dx$$

Thm: If f is integrable on D

then

$$\int_D f = \int_{x:\,D_x \neq \phi} \left(\int_{D_x} f(x,y)\,dy \right) dx$$

(red. formula)

$$D_x = \{ y : (x,y) \in D \}$$

$$= \int_{y:\,D^y \neq \phi} \left(\int_{D^y} f(x,y)\,dx \right) dy$$

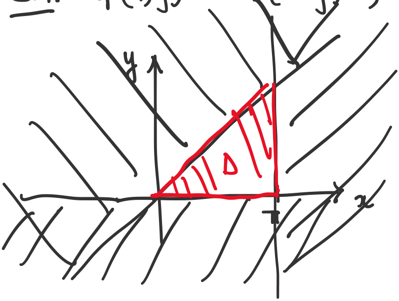
$D^y = \{x : (x,y) \in D\}$

Ex 5.3.2. Compute

$$\int_D \cos(x+y)\,dx\,dy$$

$D = \{(x,y) \in \mathbb{R}^2 : \quad 0 \leq y \leq \underline{x \leq} \pi\}$

Sol: $f(x,y) = \cos(x+y)$, $f \in \mathcal{C}(D)$

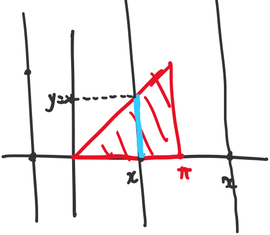


D is closed and bdd $\Rightarrow$ f is int on D.

(Thm: $f \in \mathcal{C}(D)$, D closed & bdd $\Rightarrow$ f integrable on D.)

To compute $\int_D f$ we apply RF:

$$\int_D f = \int_{x : D_x \neq \emptyset} \left(\int_{D_x} \cos(x+y)\,dy \right) dx$$



$$D_x = \begin{cases} \emptyset & x<0 \\ [0,x] & \boxed{0\le x\le \pi} \\ \emptyset & x>\pi \end{cases}$$

$$= \int_{0\le x\le \pi} \overline{\left(\boxed{\int_0^x} \cos(x+y)\,dy \right)}\, dx$$

$$\left(\partial_y(\sin(x+y)) \right.$$

$$\Big[\sin(x+y)\Big]_{y=0}^{y=x}$$

$$= \int_0^{\pi} (\sin 2x - \sin x)\, dx$$

$$= \left[\underset{= 0}{-\frac{\cos 2x}{2}} + \cos x \right]_0^{\pi}$$

$$= \qquad -1-1 = -2. \quad \blacksquare$$

Do 5.7.1 # 1,2,3.