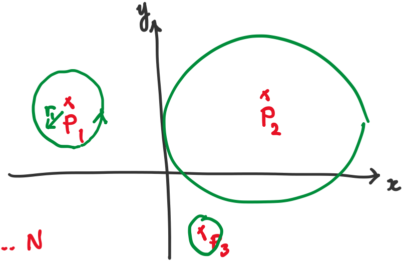
Wednesday, 13 December 2023 08:37

Thm: $\vec{F} = \vec{F}(x,y) : D = \mathbb{R}^2 \setminus \{P_1, P_2, \dots P_N\} \to \mathbb{R}^2$

If

i) $\vec{F}$ is irrot.

ii) $\oint_{\partial B(P_j, r_j]} \vec{F} = 0$



$j = 1 \dots N$

where $B(P_j, r_j] \cap \{P_1 \dots P_N\} = \{P_j\}$

$(\; P_i \notin B(P_j, r_j] \quad i \neq j \;)$

$\Rightarrow \vec{F}$ is conservative.

In part: if $\vec{F} : D = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is irrot

$\Rightarrow \vec{F}$ is cons.

Example: $\vec{F}(x,y) = \left(- \frac{y}{x^2+y^2} , \frac{x}{x^2+y^2} \right)$

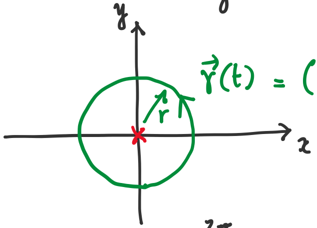
defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$

This $\vec{F}$ is irrot but not cons.

Sol: We already verified that $\vec{F}$ is irrot. $(\partial_y F_1 = \partial_x F_2)$

To verify that $\vec{F}$ is not cons we will show that $\exists \vec{\gamma}$ circuit s.t.

$$\oint_{\vec{\gamma}} \vec{F} \neq 0$$



$\vec{\gamma}(t) = (r\cos t, r\sin t) \quad t \in \underline{[0, 2\pi]}$

$$\oint_{\vec{\gamma}} \vec{F} = \int_0^{2\pi} \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt$$

$$\vec{F}(\vec{\gamma}(t)) = \vec{F}(r\cos t, r\sin t)$$

$$= \left(-\frac{r\sin t}{\underbrace{(r\cos t)^2 + (r\sin t)^2}_{\substack{r^2(\cos^2 t + \sin^2 t) \\ \parallel \\ 1}}}, \frac{r\cos t}{\parallel} \right)$$

$$= \left(-\frac{\cancel{r}\sin t}{r^{\cancel{2}}}, \frac{\cancel{r}\cos t}{r^{\cancel{2}}}\right)$$

$$= \frac{1}{r}\left(-\sin t, \cos t\right)$$

$$\vec{\gamma}'(t) = \left(-r\sin t, r\cos t\right)$$

$$\Rightarrow \oint_{\vec{\gamma}} \vec{F} = \int_0^{2\pi} \underbrace{\frac{1}{\cancel{r}}\left(-\sin t, \cos t\right)}_{\vec{F}(\vec{\gamma}(t))} \cdot \underbrace{\left(-\cancel{r}\sin t, \cancel{r}\cos t\right)}_{\vec{\gamma}'(t)} \, dt$$

$$= \int_0^{2\pi} \underset{\overset{\parallel}{1}}{\left(-\sin t\right)^2 + \left(\cos t\right)^2} \, dt$$

$$= \int_0^{2\pi} 1 \, dt = 2\pi \neq 0 .$$

■

Ex 4.4.9

$$\vec{F}(x,y) := \left(\frac{ax+by}{x^2+y^2}, \frac{cx+dy}{x^2+y^2}\right)$$

on $D = \mathbb{R}^2 \setminus \{\vec{0}\}.$

i) det a, b, c, d s.t $\vec{F}$ is irrot

ii) 〃 〃 〃 $\vec{F}$ is cons

iii) in case ii), det all the pots of $\vec{F}$

Sol: i) $\vec{F}$ irrot $\Leftrightarrow$ $\partial_y F_1 \equiv \partial_x F_2$ on D

$$\partial_y F_1 = \partial_y \frac{\boxed{ax+by}}{x^2+y^2} = \frac{b\cdot(x^2+y^2) - (ax+by)2y}{(x^2+y^2)^2}$$

$$= \frac{bx^2 + \overbrace{\cancel{by^2} - 2axy - \cancel{2}by^2}}{(x^2+y^2)^2}$$

$$\partial_x F_2 = \partial_x \frac{\boxed{cx+dy}}{x^2+y^2} = \frac{\underline{c(x^2+y^2)} - (\underline{cx}+dy)\underline{2x}}{(x^2+y^2)^2}$$

$$= \frac{-cx^2 + cy^2 - 2dxy}{(x^2+y^2)^2}$$

$\Rightarrow$ $\partial_y F_1 \equiv \partial_x F_2$ on D $\Leftrightarrow$

$$\underbrace{bx^2} - \underbrace{2axy} - \underbrace{by^2} \equiv \underbrace{-cx^2} + \underbrace{cy^2} - \underbrace{2dxy}$$

$\forall (x,y) \in \mathbb{R}^2 \xcancel{\setminus \{\vec{0}\}}$

$$\iff \begin{cases} b = -c \\ -2a = -2d \\ -b = c \end{cases}$$

$$\iff \begin{cases} b = -c \\ a = d \end{cases} \qquad \begin{cases} c = -b \\ d = a \end{cases}$$

$\vec{F}$ is irrot $\iff (*)\ \vec{F} = \left(\frac{ax+by}{x^2+y^2} , \frac{-bx+ay}{x^2+y^2} \right)$

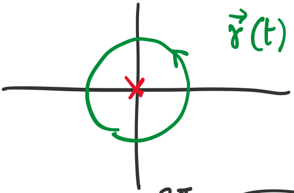
ii) To be cons $\vec{F}$ must be irrot

$\Rightarrow \vec{F}$ has to be found among $(*)$

Since $\vec{F}$ is def'd on $\mathbb{R}^2 \setminus \{\vec{0}\}$, it is sufficient to verify

$$\oint_{\partial B(0,r]} \vec{F} = 0$$

$r = 1$



$$\vec{\gamma}\,'(t) = (-\sin t, \cos t)$$

$$\vec{\gamma}(t) = (\cos t, \sin t)$$

$$t \in [0, 2\pi]$$

$$\oint_{\vec{\gamma}} \vec{F} = \int_0^{2\pi} \underbrace{\left(\frac{a\cos t + b\sin t}{1}, \frac{-b\cos t + a\sin t}{1} \right)}_{\vec{F}(\vec{\gamma}(t))} \cdot \underbrace{(-\sin t, \cos t)}_{\vec{\gamma}\,'(t)}$$

$$= \int_0^{2\pi} -a\cancel{\cos t}\sin t - b\sin^2 t - b\cos^2 t + a\cancel{\sin t}\cos t\, dt$$

$$-b(\sin^2 + \cos^2)$$

$$= -b$$

$$= \int_0^{2\pi} -b\, dt = -b \cdot 2\pi$$

$$\Rightarrow \quad \oint_{\vec{\gamma}} \vec{F} = 0 \iff b = 0$$

$\vec{F}$ is cons $\Leftrightarrow$ $\vec{F} = \left(\frac{ax}{x^2+y^2}, \frac{ay}{x^2+y^2} \right)$

iii) For $\vec{F}$ found at ii) we det a pot f by solving

$$\nabla f = \vec{F}$$

$$\Leftrightarrow (S) \begin{cases} \boxed{\partial_x f = \frac{ax}{x^2+y^2}} \\ \partial_y f = \frac{ay}{x^2+y^2}. \end{cases}$$

$$\Rightarrow f(x,y) = \int \boxed{\frac{ax}{x^2+y^2}} \, dx + c(y)$$

$$= \frac{a}{2} \int \frac{2x}{x^2+y^2} \, dx + c(y)$$

$$f(x,y) = \frac{a}{2} \log(x^2+y^2) + c(y)$$

Now, this f verifies the 2nd cond of

system (S) $\Longleftrightarrow$

$$\frac{a}{\cancel{2}} \cdot \cancel{\frac{1}{x^2+y^2}} \cdot \cancel{2}y + c'(y) = \cancel{\frac{ay}{x^2+y^2}}\ 0$$

$$\Longleftrightarrow \quad c'(y) = 0 \quad \Longleftrightarrow \quad c \equiv c_0$$

$\Longrightarrow$ all the pots for $\vec{F}$ are

$$f(x,y) = \frac{a}{2}\log(x^2+y^2) + c_0 , \quad c_0 \in \mathbb{R}.$$

$\blacksquare$

Do Ex 4.4.10/11/12/13

4.4.14

$$\vec{F}(x,y,z) = \left(\overbrace{\frac{1}{x} + \frac{y^{\alpha}}{1+x^2y^2}}^{F_1}, \overbrace{\boxed{\frac{1}{y} + \frac{x}{1+x^2y^2}}}^{F_2}, \overbrace{\frac{1}{z}}^{F_3} \right)$$

$(x,y,z) \in D = \underline{]0,+\infty[}^{3}$

i) find α s.t. $\vec{F}$ is irrot

ii) for α found in i), say whether $\vec{F}$ is cons
or not and, in aff. case, det pots of $\vec{F}$

i) $\vec{F}$ irrot $\iff$ $\vec{F} = (F_1, F_2, F_3)$

$$\begin{cases} \partial_y F_1 = \partial_x F_2 \leftarrow \\ \partial_z F_1 = \partial_x F_3 \quad \text{on } D \\ \partial_z F_2 = \partial_y F_3 \end{cases}$$

$$\iff \begin{cases} \dfrac{\alpha y^{\alpha-1}(1+x^2y^2) - y^{\alpha} 2x^2 y}{(1+x^2y^2)^2} = \dfrac{1\cdot(1+x^2y^2) - x\,2xy^2}{(1+x^2y^2)^2} \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\iff \alpha y^{\alpha-1} + \alpha x^2 y^{\alpha+1} - 2x^2 y^{\alpha+1} = 1 + x^2y^2 - 2x^2y^2$$

$$\boxed{\alpha y^{\alpha-1} + (\alpha-2)x^2 y^{\alpha+1} = \textcircled{1} - x^2y^2}$$

$$\Rightarrow \boxed{\alpha = 1}$$

$\vec{F}$ irrot $\iff$ $\vec{F} = \left(\dfrac{1}{x} + \dfrac{y}{1+x^2y^2}, \dfrac{1}{y} + \dfrac{x}{1+x^2y^2}, \dfrac{1}{z}\right)$

ii) $\vec{F}$ is cons $\Leftrightarrow$ $\exists f : \nabla f = \vec{F}$

$(f = f(x,y,z))$

$$\Leftrightarrow \begin{cases} \partial_x f = \frac{1}{x} + \frac{y}{1+x^2y^2} \\ \partial_y f = \frac{1}{y} + \frac{x}{1+x^2y^2} \\ \partial_z f = \frac{1}{z} \end{cases}$$

$$\hookrightarrow f(x,y,z) = \int \frac{1}{z}\, dz + c(x,y)$$

$$= \log z + \boxed{c(x,y)}$$

Plugging this into the first eqn

$$\partial_x f = \partial_x c = \frac{1}{x} + \frac{y}{1+x^2y^2}$$

$$\Rightarrow \underline{c(x,y)} = \int \frac{1}{x} + \frac{y}{1+x^2y^2}\, dx + k(y)$$

$$= \log x + \int \frac{y}{1+(xy)^2}\, dx + k(y)$$

$$\partial_x \operatorname{arctg}(xy)$$

$$= \frac{1}{1+(xy)^2} \cdot y$$

$$= \log x + \underbrace{\text{arctg}(xy) + k(y)}_{\downarrow}$$

$$\Rightarrow f(x,y,z) = \log z + \log x + \text{arctg}(xy) + k(y)$$

Plugging this into the 2nd eqn of system $\nabla f = \vec{F}$ we get

$$\partial_y f = \cancel{\frac{1}{1+(xy)^2}}\, x + k'(y) = \frac{1}{y} + \cancel{\frac{x}{1+x^2y^2}}$$

$$\Leftrightarrow \quad k'(y) = 1/y \quad \Leftrightarrow \quad k(y) = \log y + k_0$$

Conclusion: the pots of f are

$$\boxed{f(x,y,z) = \log(xyz) + \text{arctg}(xy) + k_0 \qquad k_0 \in \mathbb{R}}$$

$\blacksquare$

Do 4.4.15/16/17.

Multiple Integration

Goal: define the operation of integral for functs

$$f = f(x_1, \dots, x_m)$$

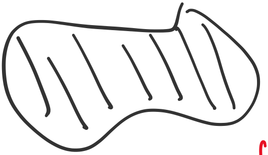
where $f : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}$

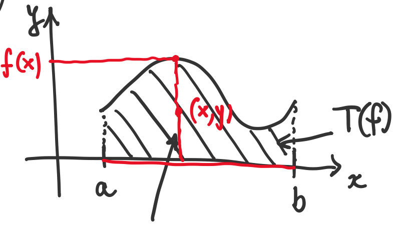
$$\int_D f(x_1, \dots x_m)\, dx_1 \dots dx_m$$

A quick review of 1-var integrals.

Integral solve a geom. pb: def. a gen. method to compute the area of a plane figure.

$f : [a,b] \longrightarrow [0,+\infty[$

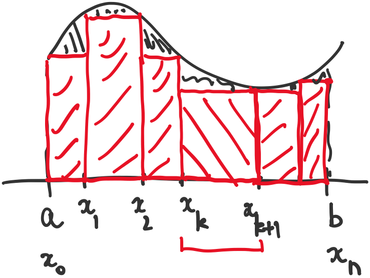




$$T(f) = \{(x,y) : a \leqslant x \leqslant b,\ \ 0 \leqslant y \leqslant f(x)\}$$

Riemann int provided a method (under suit. conds on f) to define

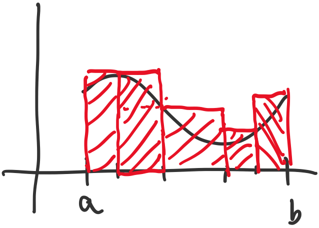
Area $(T(f)) \equiv A(T(f))$.



$$\pi = \{\overset{a}{\overset{\|}{x_0}} < x_1 < x_2 < x_n = b\}$$

$$\underline{S}(\pi) = \sum_{k=0}^{n-1} (x_{k+1} - x_k)\, \underset{\substack{\| \\ \inf\limits_{[x_k, x_{k+1}]} f(x)}}{h_k} \leqslant \text{“}A(T(f))\text{”}$$

$$\operatorname{Sup}\{\underline{S}(\pi) : \pi \text{ sub divisions of } [a,b]\} =: \boxed{\underline{A}(T(f))}$$



$$\bar{S}(\pi) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot \underset{\substack{\| \\ \sup_{[x_k, x_{k+1}]} f(x)}}{H_k} \geq \text{"} A(T(f)) \text{"}$$

$$\boxed{\bar{A}(T(f))} := \operatorname{Inf} \left\{ \bar{S}(\pi) : \pi \text{ subdiv.} \right\}$$

In gen.

$$\underline{A}(T(f)) \leq \bar{A}(T(f))$$

and it can be $<$

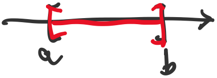
(ex : $f = 1_{\mathbb{Q}^c \cap [0,1]}$)

Def: If $\underline{A}(T(f)) = \bar{A}(T(f))$

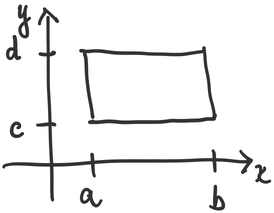
we call the common value $A(T(f))$.

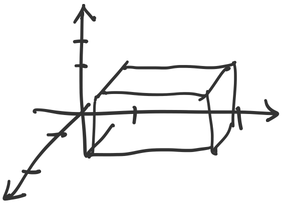
Thm: If $f \in \mathcal{C}([a,b])$, $f \geq 0 \Rightarrow \exists A(T(f))$. $\lrcorner$

Let's now cons $f = f(x,y)$



$I = \{x \in \mathbb{R} : a \leq x \leq b\}$

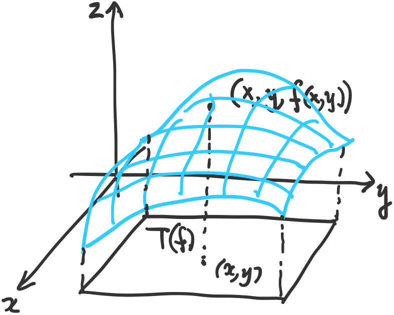




Def: An interval in $\mathbb{R}^m$ is a set

$$\{(x_1 \text{—} x_m) \in \mathbb{R}^m : a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2,\ \cdots\ a_m \leq x_m \leq b_m\}$$

$= [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m]$



$$T(f) = \{ (x,y,z) \in \mathbb{R}^3 : (x,y) \in I , 0 \leq z \leq f(x,y) \}$$

Goal: def Volume of T(f).