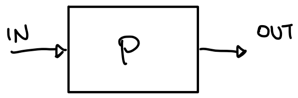
COMPUTABILITY (12/12/2023)

Rice - Shapiro's Theorem



properties of I/O behavior

dual view

① properties of functions computed by program $\mathcal{A} \subseteq \mathcal{C}$

$\mathcal{T} = \{ f \in \mathcal{C} \mid f \text{ is total} \} = \{ f \in \mathcal{C} \mid dom(f) = \mathbb{N} \}$

$ONE = \{ \mathbb{1} \}$

$\vdots$

② saturated / extensional property of programs $A \subseteq \mathbb{N}$

$T = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{T} \}$

$P_{ONE} = \{ x \in \mathbb{N} \mid \varphi_x = \mathbb{1} \}$

$\vdots$

→ <u>Rice's Theorem</u> : only trivial (true / false) extensional properties are decidable

→ <u>Rice - Shapiro's Theorem</u> : an extensional property of programs $\boxed{\text{can be}}$ semidecidable only when it is <u>finitary</u>

(behaviour of the program on a finite amount of inputs)

Rice - Shapiro's Theorem

Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable function

and let $A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

If $\boxed{A \text{ is r.e.}}$ $(*)$ then

$$\boxed{\forall f \ (f \in \mathcal{A} \iff \exists \vartheta \subseteq f, \ \vartheta \text{ finite } \vartheta \in \mathcal{A})} \quad (**)$$

proof

We want to show that $(*) \Rightarrow (**)$

For this we show $\neg (**) \Rightarrow \neg (*)$

This amounts to

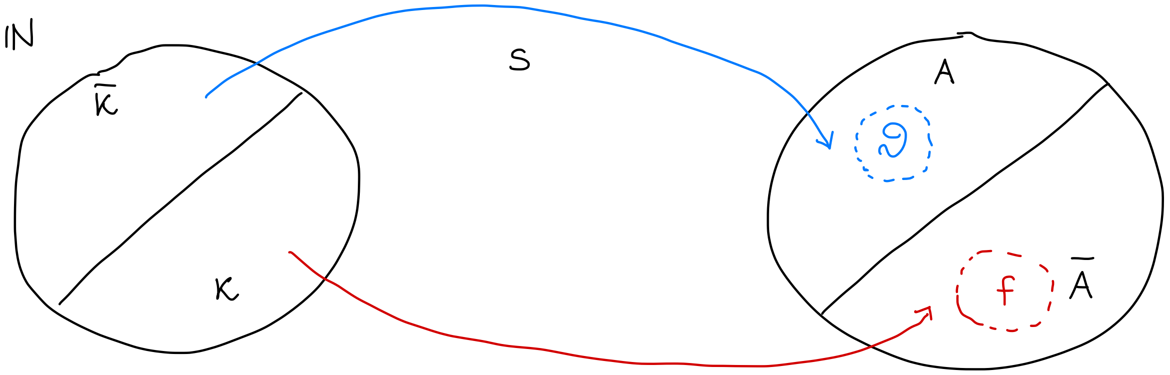
① $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$, ϑ finite $\vartheta \in \mathcal{A}$ $\Rightarrow$ A not r.e.

② $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite $\vartheta \notin \mathcal{A}$ $\Rightarrow$ A not r.e.

① $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$, ϑ finite $\vartheta \in \mathcal{A}$ $\Rightarrow$ A not r.e.

Let $\underline{f \notin \mathcal{A}}$ be s.t there is $\vartheta \subseteq f$ ϑ finite $\underline{\vartheta \in \mathcal{A}}$

$$\bar{K} = \underbrace{\{ x \in \mathbb{N} \mid \varphi_x(x)\uparrow \}}_{\text{not r.e.}} \leq_m A$$



Define

$$g(x,y) = \begin{cases} \vartheta(y) & \text{if } x \in \bar{K} \\ f(y) & \text{if } x \in K \end{cases} \quad \overset{\subseteq f}{}$$

$$= \begin{cases} \vartheta(y) = f(y) & \text{if } x \in \bar{K} \text{ and } y \in dom(\vartheta) \\ \uparrow & \text{if } x \in \bar{K} \text{ and } y \notin dom(\vartheta) \\ f(y) & \text{if } x \in K \end{cases}$$

$$= \begin{cases} f(y) & \text{if } \underbrace{x \in K \text{ or } y \in dom(\vartheta)} \\ \uparrow & \text{otherwise} \end{cases}$$

$$Q(x,y) = \underbrace{\text{“} \underbrace{x \in K}_{\text{semidec.}} \text{ or } \underbrace{y \in \overbrace{dom(\vartheta)}^{\text{finite}}}_{\text{decidable}} \text{”}}_{\text{semi decidable}}$$

$$sc_Q(x,y) = \begin{cases} 1 & \text{if } Q(x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

computable

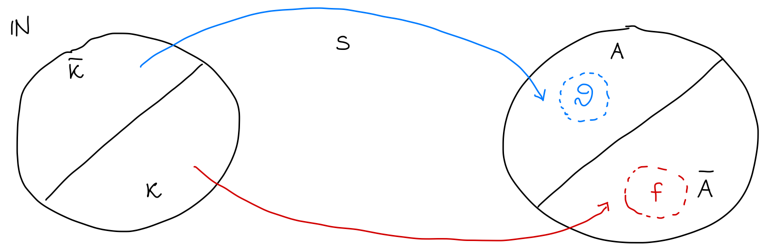
$$= f(y) \cdot sc_Q(x,y)$$

computable by composition

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t. $\forall x,y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} \vartheta(y) & \text{if } x \in \bar{K} \\ f(y) & \text{if } x \in K \end{cases}$$

We show that s is the reduction function for $\bar{K} \leq A$



x if $x \in \bar{K}$ then $s(x) \in A$

let $x \in \bar{K}$. Then

$$\varphi_{s(x)}(y) = \vartheta(y) \qquad \forall y$$

This means

$$\varphi_{s(x)} = \vartheta \in \mathcal{A}$$

hence $s(x) \in A$

* if $x \in K$ then $s(x) \in \bar{A}$

if $x \in K$ then

$$\varphi_{s(x)}(y) = f(y) \quad \forall y$$

then

$$\varphi_{s(x)} = f \notin \mathcal{A}$$

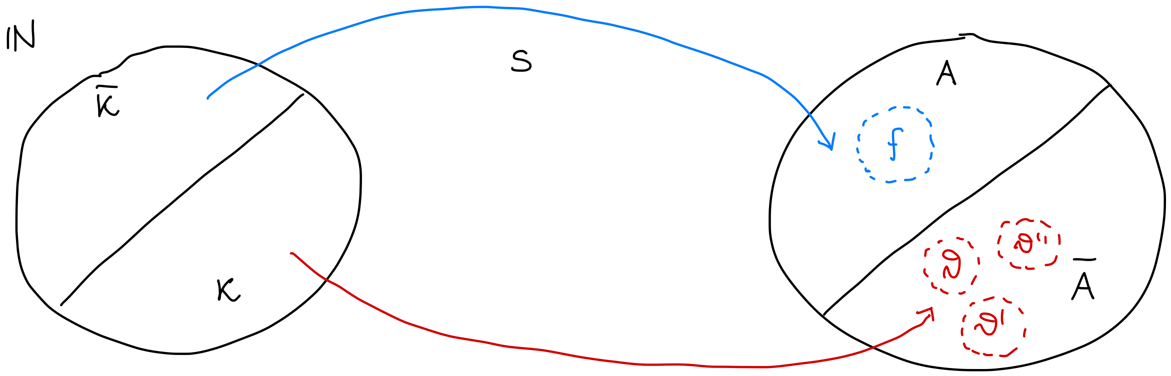
and thus $s(x) \notin A$.

Hence $\bar{K} \leq_m A$ and since $\bar{K}$ not r.e. we conclude A not r.e.

(2) $\exists f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite $\vartheta \notin \mathcal{A}$ then A not r.e.

let $f \in \mathcal{A}$ and assume all $\vartheta \subseteq f$, ϑ finite $\vartheta \notin \mathcal{A}$

We show $\bar{K} \leq_m A$



$\vartheta, \vartheta', \vartheta'' \subseteq f$

We define

$$g(x,y) = \begin{cases} f(y) & \text{if } ``x \in \bar{K}" \quad P_x(x)\uparrow \\ \underset{\text{some } \vartheta \subseteq f \text{ finite}}{\vartheta(y)} & \text{if } x \in K \quad P_x(x)\downarrow \end{cases}$$

$$= \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

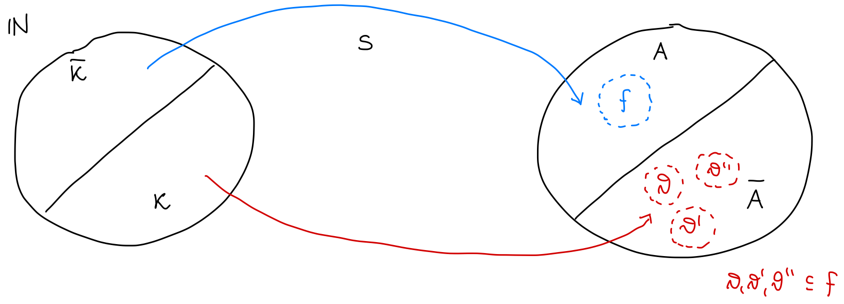
$$= \underbrace{f(y) + \mu z. \underbrace{\chi_H(x,x,y)}_{\substack{0 \text{ if } \neg H(x,x,y) \\ 1 \text{ if } H(x,x,y)}}}_{\substack{0 \text{ if } \neg H(x,x,y) \\ \uparrow \text{ otherwise}}}$$

computable

By smn theorem there $s : \mathbb{N} \to \mathbb{N}$ total computable s.t. $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

We show that s is a reduction function for $\bar{K} \leq_m A$



* if $x \in \bar{K}$ then $s(x) \in A$

if $x \in \bar{K}$ then $\varphi_x(x)\uparrow$ i.e. $P_x(x)\uparrow$.

Then $\forall y$ $\neg H(x,x,y)$. Thus $\forall y$ $\varphi_{s(x)}(y) = f(y)$.

Therefore $\varphi_{s(x)} = f \in \mathcal{A}$ and thus $s(x) \in A$.

* if $x \in K$ then $s(x) \in \bar{A}$

if $x \in K$ then $\varphi_x(x)\downarrow$ i.e. $P_x(x)\downarrow$

Then $\exists y_0 \in \mathbb{N}$ s.t. $\forall y < y_0$ $\neg H(x,x,y)$

$\forall y \geq y_0$ $H(x,x,y)$

thus

$$\varphi_{s(x)}(y) = \begin{cases} f(y) & \text{if } y < y_0 \\ \uparrow & \text{otherwise} \end{cases}$$

Then $\varphi_{s(x)} \subseteq f$

$dom(\varphi_{s(x)}) \subseteq [0, y_0)$ finite

i.e. $\varphi_{s(x)}$ is a finite subfunction of f, hence $\varphi_{s(x)} \in \bar{\mathcal{A}}$

hence $s(x) \in \bar{A}$.

Thus $\bar{K} \leq_m A$ and, since $\bar{K}$ not r.e., A not r.e.

$\square$

Typical use of Rice – Shapiro : Prove that $A \subseteq \mathbb{N}$ not r.e.

by showing either (1) or (2)

(1) $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$, ϑ finite $\vartheta \in \mathcal{A}$

(2) $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite $\vartheta \notin \mathcal{A}$

<u>Exercise</u> : Totality $\quad \mathcal{T} = \{ f \in \mathcal{C} \mid f \text{ is total} \}$

$$T = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{T} \}$$

* <u>T is not r.e.</u>

  $id \in \mathcal{T} \qquad dom(id) = \mathbb{N}$

  $\forall \vartheta \subseteq id$, ϑ finite $\quad dom(\vartheta)$ finite $\neq \mathbb{N} \quad \Rightarrow \vartheta \notin \mathcal{T}$

  $\Rightarrow$ T is not r.e. by Rice - Shapiro

* <u>$\overline{T}$ is not r.e.</u>

  $id \notin \overline{\mathcal{T}}$ and $\vartheta = \emptyset \subseteq id \qquad \vartheta \in \overline{\mathcal{T}}$

  $\emptyset(x)\uparrow$
  $\forall x$

  $\Rightarrow \overline{T}$ is not r.e.

* <u>Exercise</u> : $ONE = \{ x \mid \varphi_x = \mathbb{1} \}$

  $\varphi_x \in \{\mathbb{1}\}$

* <u>ONE is not r.e.</u>

  $\mathbb{1} \in \{\mathbb{1}\}$ and $\forall \vartheta \subseteq \mathbb{1}$, ϑ finite $\vartheta \notin \{\mathbb{1}\}$

  hence by Rice - Shapiro ONE not r.e.

* <u>$\overline{ONE}$ is not r.e.</u>

  $\mathbb{1} \notin \overline{\{\mathbb{1}\}}$ and $\vartheta = \emptyset \subseteq \mathbb{1} \quad \vartheta \in \overline{\{\mathbb{1}\}}$ then $\overline{ONE}$ not r.e.

<u>OBSERVATION</u> : The converse implication of Rice-Shapiro is false

$\mathcal{A} \subseteq \mathcal{C} \qquad A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

$\forall f \quad (f \in \mathcal{A} \iff \exists \vartheta \subseteq f,\ \vartheta \text{ finite},\ \vartheta \in \mathcal{A})$

$$\cancel{\Downarrow}$$

A r.e.

<u>counter example</u> :

$\mathcal{A} \subseteq \mathcal{C}$

(a) $\forall f \quad (f \in \mathcal{A}$ iff $\exists \vartheta \subseteq f,\ \vartheta$ finite $\ \vartheta \in \mathcal{A})$

(b) $A = \{ x \mid \varphi_x \in \mathcal{A} \}$ not r.e.

we claim $\qquad \mathcal{A} = \{ f \mid dom(f) \cap \overline{K} \neq \emptyset \}$

satisfies (a) and (b)

(a) let f be a function

* $f \in \mathcal{A} \quad \leadsto \quad dom(f) \cap \overline{K} \neq \emptyset$

i.e. $\exists x_0 \in dom(f) \cap \overline{K}$

if we define

$$\vartheta(x) = \begin{cases} f(x) & x = x_0 \\ \uparrow & \text{otherwise} \end{cases}$$

$\vartheta \subseteq f$ finite $\qquad dom(\vartheta) = \{x_0\}$

$\leadsto \ dom(\vartheta) \cap \overline{K} = \{x_0\} \neq \emptyset$

* if there is $\vartheta \subseteq f$, ϑ finite $\ \vartheta \in \mathcal{A}$

$$\underset{\substack{\cap \\ dom(f)}}{dom(\vartheta)} \cap \overline{K} \neq \emptyset$$

$\leadsto \quad dom(f) \cap \overline{K} \supseteq dom(\vartheta) \cap \overline{K} \neq \emptyset$

$\leadsto \quad dom(f) \cap \overline{K} \neq \emptyset \quad \leadsto \quad f \in \mathcal{A}$.

(b) $A = \{ x \mid \varphi_x \in \mathcal{A} \} = \{ x \mid dom(\varphi_x) \cap \overline{K} \neq \emptyset \}$ not r.e.

assume that you can semidecide if $x \in A$

$\rightsquigarrow$

in order to check if $x \in \overline{K}$ create

$$\boxed{\begin{array}{l} \text{def } P(y): \\ \quad \text{if } y = x \\ \qquad \text{return } 0 \\ \quad \text{else loop} \end{array}}$$

and check if $\underbrace{dom(P)}_{= \{x\}} \cap \overline{K} \neq \emptyset$?

we show $\overline{K} \leq_m A$

define $g(x,y) = \mu z. |x-y| = \begin{cases} 0 & \text{if } x = y \\ \uparrow & \text{otherwise} \end{cases}$ computable

$= \varphi_{s(x)}(y)$

by smn

there is $s: \mathbb{N} \to \mathbb{N}$ total computable

s is the reduction function for $\overline{K} \leq_m A$

$$x \in \overline{K} \iff \underbrace{dom(\varphi_{s(x)})}_{= \{x\}} \cap \overline{K} \neq \emptyset \iff s(x) \in A$$

hence A is not r.e.