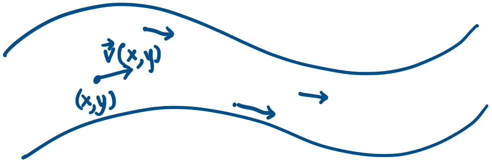
Thursday, 7 December 2023 08:32

Vector Fields

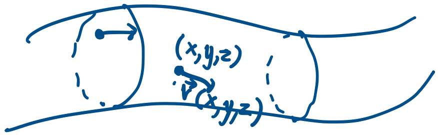
A vector field is a function

$$\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

Vector fields are used to model important physical objects:

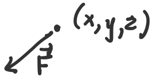


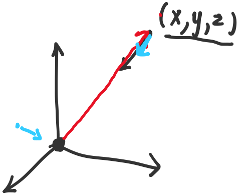
$$\vec{v} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$



$$\vec{v} : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

Force field





$$\boxed{-\frac{(x,y,z)}{\|(x,y,z)\|}} = -\vec{n}(x,y,z)$$

$$\vec{G}(x,y,z) = -\underline{\vec{n}(x,y,z)}\;\frac{\overset{k\,\textcircled{M}}{\overset{\swarrow}{C}}}{\|(x,y,z)\|^2}$$

$$\boxed{\vec{G}(x,y,z) = -kM\,\frac{(x,y,z)}{\|(x,y,z)\|^3}}$$

Def: Given a vector field $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ a potential of $\vec{F}$ is a function

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

s.t.

$$f' = \vec{F}$$

$$\Longleftrightarrow \quad \boxed{\nabla f = \vec{F}}$$

Pb: Under which conds a vector field $\vec{F}$ has a pot. f ? And, in this case, how can we compute such f ?

Rmk: If $d = 1$, $F : D \subset \mathbb{R} \to \mathbb{R}$, a pot f is s.t

$$f'(x) = F(x)$$

In other words, a pot of F is nothing but a primitive of F, we denote it by

$$f(x) = \int F(x)\,dx$$

What we know about this pb is:

- if D is an interval, then all pot. are
$$f(x) = \underbrace{\int F(x)\,dx}_{\text{one of the prim of } F} + c$$
- if $F \in \mathcal{C}([a,b]) \Rightarrow \exists$ always a pot of F

x

$$f(x) = \int_a F(y)\,dy \qquad \text{(Fund Thm of Int. Calc.)}$$

$$(\ \exists\ f'(x) = F(x)\ )$$

∎

In gen, if $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$$\nabla f = \boxed{\vec{F}}$$

can be written in the following way:

if

$$\begin{array}{l} \vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \dots, F_d(\vec{x})) \\ \parallel \\ \nabla f(\vec{x}) = (\partial_1 f(\vec{x}), \partial_2 f(\vec{x}), \dots, \partial_d f(\vec{x})) \end{array}$$

we have

$$\nabla f = \vec{F} \iff \begin{cases} \partial_1 f = F_1 \\ \partial_2 f = F_2 \\ \vdots \\ \partial_d f = F_d \end{cases}$$

Example: Det (if any) pot. for

$$\vec{F}(x,y) := \Big(\overset{F_1}{\boxed{y}}, \overset{F_2}{\boxed{x}}\Big) \qquad (x,y) \in D = \mathbb{R}^2$$

Sol: $f = f(x,y)$ is pot for $\vec{F}$ $\iff$

$$\nabla f = \vec{F} \iff \begin{cases} \partial_x f = y \\ \partial_y f = x \end{cases}$$

$$\iff \begin{cases} \boxed{\partial_x f(x,y) = y} \\ \partial_y f(x,y) = x \end{cases}$$

$$\boxed{\partial_x f(x,y) = y}$$

$$\Updownarrow$$

$$\boxed{f(x,y) = yx + c(y)}$$

?

$f'(x) = 3$

$3x + c$

Now

$$\boxed{\partial_y f = x} \iff \partial_y \left(yx + c(y) \right) = x$$

$$\iff \cancel{x} + c'(y) = \cancel{x}\ 0$$

$$\iff c'(y) = 0$$

$$\iff \boxed{c(y) = c}$$

So: $\nabla f = \vec{F} \iff \boxed{f(x,y) = xy + c}$

Example: Same pb with

$$\vec{F}(x,y) = (\overset{F_1}{\boxed{y}}, \overset{F_2}{\boxed{-x}}).$$

Sol: f is a pot for $\vec{F}$ $\iff$

$$\nabla f = \vec{F} \iff \begin{cases} \partial_x f = y \\ \partial_y f = -x \end{cases}$$

$$\partial_x f = y \iff f(x,y) = yx + c(y)$$

$$\partial_y f = -x \iff \partial_y \left(yx + c(y) \right) = -x$$

$$x + c'(y) = -x$$

$$\iff c'(y) = \boxed{-2x}$$

$$\iff c(y) = -2xy + c$$

$$\Rightarrow f(x,y) = -xy + c \quad ?$$

$$\partial_x f = -y \neq F_1 \quad !$$

This means that $\nexists f : \nabla f = \vec{F}$. ■

Thm (Schwarz)

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ s.t.

i) f is differentiable on D

ii) $\exists\ \partial_j(\partial_i f) \in \mathscr{C}$, $\exists\ \partial_i(\partial_j f) \in \mathscr{C}$

Then

$$\partial_j(\partial_i f) = \partial_i(\partial_j f) \text{ on } D.$$

Example: $f(x,y) = xy^2 + y$

Here

$$\partial_x f = y^2 \quad , \quad \partial_y f = 2xy + 1 \in \mathscr{C}$$

$\Rightarrow$ f is diff on $\mathbb{R}^2$

(i) checked)

$$\partial_y(\partial_x f) = 2y \in \mathscr{C}, \qquad \partial_x(\partial_y f) = 2y \in \mathscr{C}$$

(ii) checked)

and $\partial_y(\partial_x f) = 2y = \partial_x(\partial_y f)$ ∎

Rmk: $\partial_j(\partial_i f) = \partial_i(\partial_j f)$ is not true in general (there are examples for which $\partial_x(\partial_y f) \neq \partial_y(\partial_x f)$. ∎

Now assume that $\vec{F}$ has a pot f

$$\nabla f = \vec{F}$$

$$\iff \begin{cases} \partial_1 f = F_1 \\ \partial_2 f = F_2 \\ \vdots \\ \partial_d f = F_d \end{cases}$$

If F_j have $\partial_i F_j \in \mathcal{C}$ $\forall i,j$

$\Rightarrow$

$$\partial_j f = F_j$$

$$\Downarrow$$

$$\boxed{\partial_i(\partial_j f)} = \partial_i F_j \in \mathcal{C}$$

$$\parallel$$

$$\partial_j\left(\overbrace{\underbrace{\partial_i f}_{}}^{F_i}\right) = \partial_j F_i$$

$$\Rightarrow \quad \boxed{\partial_i F_j = \partial_j F_i \qquad \forall\, i,j}$$

Prop: If $\vec{F} = (F_1 \sim F_d)$ has a pot.

and $\underline{\partial_i F_j \in \mathcal{C}(D)}$ $\forall\, i,j$

$$\Rightarrow (*) \ \boxed{\partial_i F_j = \partial_j F_i \ \text{ on } D, \ \forall\, i,j}$$

Example: $\vec{F} = (\overset{F_1}{y}, \overset{F_2}{-x})$

Let's check if $\vec{F}$ verifies $(*)$

$$\partial_i F_j \overset{?}{=} \partial_j F_i \qquad \text{for } i \neq j$$

$$\partial_x F_2 \overset{?}{=} \partial_y F_1$$

$$\begin{array}{ccc} \partial_x(-x) & \overset{?}{=} & \partial_y y \\ \| & & \\ -1 & \overset{?}{=} & 1 \end{array} \qquad \text{No!}$$

$\blacksquare$

Def: We say that a field $\vec{F}$ is

- conservative if $\vec{F}$ has a pot
- irrotational if $\partial_i F_j = \partial_j F_i$ $\forall i,j \ (i \neq j)$

So

a cons field is irrot., so to be irrot is a necessary cond to be conservative,

Note irrot $\not\Leftrightarrow$ cons.

Example

$$\vec{F}(x,y) = \left(-\overset{F_1}{\frac{y}{x^2+y^2}} , \overset{F_2}{\frac{x}{x^2+y^2}} \right)$$

Show that $\vec{F}$ is irrot on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$
but $\vec{F}$ is not cons.

Sol: $\vec{F}$ is irrot $\iff \partial_x F_2 = \partial_y F_1$

$$\partial_x F_2 = \partial_x\left(\frac{x}{x^2+y^2}\right) = \frac{\overset{1}{\overset{\shortparallel}{\boxed{\partial_x x}}}\cdot(x^2+y^2) - x\cdot\overbrace{\partial_x(x^2+y^2)}^{2x}}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$\partial_y F_1 = \partial_y\left(\ominus\,\frac{y}{x^2+y^2}\right)$$

$$= -\,\frac{x^2-y^2}{(y^2+x^2)^2}$$

Concl: $\partial_x F_2 = \partial_y F_1$ on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$

$\Rightarrow \vec{F}$ is irrot.

Let's check if $\vec{F}$ is cons: $\exists f$

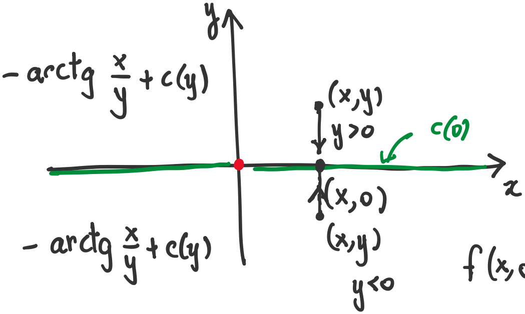
$$\nabla f = \vec{F} \iff \begin{cases} \partial_x f = -\dfrac{y}{x^2+y^2} . \\[2em] \partial_y f = \dfrac{x}{x^2+y^2} \end{cases}$$

$$\boxed{\partial_x f(x,y) = -\frac{y}{x^2+y^2}}$$

$$f(x,y) = \int -\frac{y}{x^2+y^2}\, dx + c(y)$$

$$= \begin{cases} -\displaystyle\int \frac{\cancel{y}}{\cancel{y^2}\left(\left(\frac{x}{y}\right)^2+1\right)}\, d\tfrac{x}{y} + c(y) & y \neq 0 \\[1em] \overset{=}{t} = x/y \qquad -\displaystyle\int \frac{1}{t^2+1}\, dt + c(y) & \\[1em] = -\operatorname{arctg} t + c(y) & \\[1em] = -\operatorname{arct}\left(\frac{x}{y}\right) + c(y) & \\[2em] c(0). & y = 0 \end{cases}$$

$$f(x,y) = \begin{cases} -\operatorname{arctg}\dfrac{x}{\boxed{y}} + c(y) & y \neq 0 \\ \boxed{c(0)} & y = 0 \end{cases}$$


y
$-\operatorname{arctg}\frac{x}{y} + c(y)$
(x,y)
$y>0$
$c(0)$
$(x,0)$
x
$-\operatorname{arctg}\frac{x}{y} + c(y)$
(x,y)
$y<0$

$$f(x,0) = c(0)$$

$$f(x,y) = \operatorname{arct}\left(\frac{\boxed{x}}{y}\right) + \underline{c(y)}$$

$0-$

$-\infty$

$$\xrightarrow[y\to 0-]{} -\pi/2 + c(0)$$

$$f(x,y) \xrightarrow[y\to 0+]{} +\frac{\pi}{2} + c(0)$$

f is not diff on $\{(x,0) : x \neq 0\} \subset D$

$\Rightarrow$ ∇f is not even def'd at those pts

$\Rightarrow$ $\nabla f = \vec{F}$ is not defided !

($\vec{F}$ has not a pot on D)

Notice also that

$$\partial_y f = \frac{x}{x^2+y^2} \quad ?$$

$$\partial_y f \underset{y\neq 0}{=} \partial_y\left(\mathrm{arct}\left(\frac{x}{y}\right) + c(y)\right)$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right) + c'(y)$$

$$= \frac{\cancel{y^2}}{x^2+y^2}\left(-\frac{x}{\cancel{y^2}}\right) + c'(y)$$

$$= -\frac{x}{x^2+y^2} + c'(y) \overset{?}{=} \frac{x}{x^2+y^2}$$

$$\Rightarrow \quad c'(y) = 2\,\frac{x}{x^2+y^2}$$ imposs.

(this is another justif. of non exist ■
for pot f)

Do Ex 4.4.2/3/4/5
4.4.7/4.4.9 .