Wednesday, 6 December 2023 08:41

Do Ex 3.6.17 → 25

3.6.18

$$\mathcal{M} = \{(x,y,z)\in\mathbb{R}^3 : z^2 = xy+1\}$$

Q1 $\mathcal{M} \neq \emptyset$, $\mathcal{M} = \{g = 0\}$ g subm on $\mathcal{M}$

Q2 Is $\mathcal{M}$ compact?

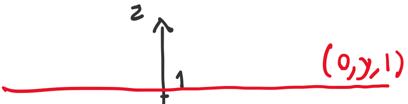
Q3 Det (if any) pts of $\mathcal{M}$ at min/max distance to $\vec{0}$.

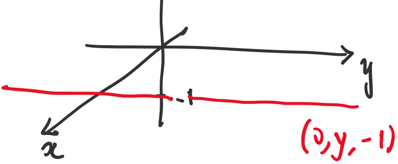
Sol:

$$\left.\begin{array}{l} (0,0,z)\in\mathcal{M} \iff z^2 = 1 \iff z = \pm 1 \\ \Rightarrow (0,0,\pm 1)\in\mathcal{M}. \end{array}\right]$$

$$(\underbrace{0,y}, z)\in\mathcal{M} \iff z^2 = 1 \iff z = \pm 1$$

$$\Rightarrow (0,y,\pm 1)\in\mathcal{M} \quad \forall y\in\mathbb{R}$$





$\mathcal{M} = \{g = 0\}$ where $g = z^2 - xy - 1$

g is submersion on $\mathcal{M} \iff \nabla g \neq \vec{0}$ on $\mathcal{M}$

$(\text{rk } \nabla g = 1)$

Now $\nabla g = \vec{0}$

$(\partial_x g = -y, \ \partial_y g = -x, \ \partial_z g = 2z$

$\in \mathcal{C}(\mathbb{R}^3) \Rightarrow g$ is diff on $\mathbb{R}^3)$

$\Updownarrow$

$$\begin{cases} -y = 0 \\ -x = 0 \\ 2z = 0 \end{cases} \iff (x, y, z) = (0,0,0) \notin \mathcal{M}$$

$\Rightarrow \nabla g \neq \vec{0}$ on $\mathcal{M}$. ✓

Q2. $\mathcal{M}$ compact $\iff \mathcal{M}$ closed & bded

$\mathcal{M}$ is closed because $\mathcal{M} = \{g = 0\}$ and $g \in \mathcal{C}$.

However, $\mathcal{M}$ is unbded: since $(0, y, \pm 1) \in \mathcal{M}$ $\forall y$

$$\| (0, y, \pm 1) \| = \sqrt{y^2 + 1} \xrightarrow[y \to \pm\infty]{} +\infty$$

$$\left(\nexists\, C : \quad \| (0, y, \pm 1) \| \leq C \quad \forall y \right)$$

Q3 $\quad \operatorname{dist}\left((x,y,z), \vec{0} \right) = \| (x,y,z) - \vec{0} \|$

$$= \| (x,y,z) \|$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$=: f(x,y,z)$$

we have to discuss

$$\min_{\mathcal{M} = \{g = 0\}} / \max f .$$

$$\boxed{\nabla f = \lambda \nabla g}$$

$$\nabla f = \left(\frac{\cancel{2}x}{\cancel{2}\underset{\sim\sim\sim}{\sqrt{x^2+y^2+z^2}}}, \frac{y}{\underset{\sim\sim\sim}{\sqrt{}}}, \frac{z}{\underset{\sim\sim}{\sqrt{}}} \right)$$

Trick: $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ is min/max

$$\Longleftrightarrow \underbrace{x^2+y^2+z^2}_{f(x,y,z)} \quad \text{"} \quad \text{"}$$

We have to solve

$$\min_{\mathcal{M} = \{g=0\}}/\max \underline{(x^2+y^2+z^2)}.$$

Existence: $f = x^2+y^2+z^2 \in \mathcal{C}(\mathbb{R}^3) \Rightarrow f \in \mathcal{C}(\mathcal{M})$,

$\mathcal{M}$ is closed & unbdd and since

$$\lim_{(x,y,z) \to \infty_3} f = +\infty$$

$\Rightarrow$ there is no max for f on $\mathcal{M}$ but $\exists \min_{\mathcal{M}} f$.

Search: Let $(x,y,z) \in \mathcal{M}$ be a min pt for f. By Q1, $\mathcal{M} = \{g=0\}$ and g is a submersion on $\mathcal{M}$. Then, According to Lagr. thm, at min pt (x,y,z)

$$\boxed{\nabla f = \lambda \nabla g}$$

$$\Updownarrow$$

$$\boxed{\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2}$$

$$\Updownarrow$$

$$\operatorname{rk}\begin{bmatrix} 2x & 2y & 2z \\ -y & -x & 2z \end{bmatrix} < 2$$

$$\Updownarrow$$

all 2×2 subdets $= 0$

$$\Leftrightarrow \begin{cases} \det \begin{bmatrix} 2x & 2y \\ -y & -x \end{bmatrix} = 0 \\ \det \begin{bmatrix} 2y & 2z \\ -x & 2z \end{bmatrix} = 0 \\ \det \begin{bmatrix} 2x & 2z \\ -y & 2z \end{bmatrix} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -\cancel{2}x^2 + \cancel{2}y^2 = 0 \\ 2\cancel{4}yz + \cancel{2}xz = 0 \\ 2\cancel{4}xz + \cancel{2}yz = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (y-x)(y+x) = 0 \\ z(2y+x) = 0 \\ z(2x+y) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ \boxed{\cancel{3}zx = 0} \\ \cancel{3zx = 0} \end{cases} \vee \begin{cases} y = -x \\ -zx = 0 \\ \cancel{-zx = 0} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} y = x \\ z = 0 \end{cases} \vee \begin{cases} y = x \\ x = 0 \end{cases} \vee \begin{cases} y = -x \\ z = 0 \end{cases} \vee \begin{cases} y = -x \\ x = 0 \end{cases}$$

$(x,x,0)$ $x \in \mathbb{R}$ $(0,0,z)$ $z \in \mathbb{R}$ $(x,-x,0)$ $x \in \mathbb{R}$ $\cancel{(0,0,z) \; z \in \mathbb{R}}$

$$z^2 = xy + 1$$

$(x,x,0) \in M \iff 0 = x^2 + 1$ no sols

$(0,0,z) \in M \iff z^2 = 1 \Rightarrow (0,0,\pm 1)$

$(x,-x,0) \in M \iff 0 = -x^2 + 1 \iff x^2 = 1$

$\iff x = \pm 1$

$\Rightarrow (1,-1,0), (-1,1,0)$

$f(0,0,\pm 1) = 1$

$f(1,-1,0) = f(-1,1,0) = 2$

Concl: $(0,0,\pm 1)$ are pts of M at min dist to $\vec{0}$. ■

Ex 3.6.17

$$\mathcal{M} = \{(x,y,z)\in\mathbb{R}^3 : \ z^2 = x^2+y^2+1, \ z = 2x^2+y^2\}$$

Q1 $\mathcal{M} \neq \phi$, is the zero set of a subm.

Q2 Is $\mathcal{M}$ compact?

Q3 Det (if any) pts of $\mathcal{M}$ at min/max quote (z).

Sol: Q1

$$(x,y,z)\in\mathcal{M} \iff \begin{cases} z^2 = x^2+y^2+1 \\ z = 2x^2+y^2 \end{cases}$$

$$(0,y,z)\in\mathcal{M} \iff \begin{cases} z^2 = y^2+1 \\ z = y^2 \end{cases} \implies \begin{cases} z^2 = z+1 \\ z = y^2 \end{cases}$$

$$\iff \begin{cases} z^2 - z - 1 = 0 \\ y^2 = z = \frac{1\pm\sqrt{5}}{2} \end{cases} \qquad z = \frac{+1\pm\sqrt{5}}{2}$$

$$\Leftrightarrow \begin{cases} z = \dfrac{1+\sqrt{5}}{2} \\ y^2 = \dfrac{1+\sqrt{5}}{2} \end{cases} \Rightarrow y = \pm\sqrt{\dfrac{1+\sqrt{5}}{2}}$$

$$\left(0, \pm\sqrt{\frac{1+\sqrt{5}}{2}}, \frac{1+\sqrt{5}}{2}\right) \in \mathcal{M}$$

$$\Rightarrow \mathcal{M} \neq \phi.$$

$\mathcal{M} = \{ g_1 = 0, \ g_2 = 0 \}$ where

$$g_1 = x^2 + y^2 - z^2 + 1$$

$$g_2 = 2x^2 + y^2 - z$$

(clearly g_1, g_2 are diff)

$$\mathcal{M} = \{ \vec{g} = \vec{0} \} \qquad \vec{g} = (g_1, g_2).$$

$\vec{g}$ is submersion on $\mathcal{M}$ $\Leftrightarrow$

$$\operatorname{rk}[\vec{g}\,'] = 2 \quad \text{on } \underline{\mathcal{M}}.$$

$$\shortparallel$$

$$\operatorname{rk}\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix}$$

Now

$$\mathrm{rk}\begin{bmatrix}\nabla g_1\\ \nabla g_2\end{bmatrix} = \mathrm{rk}\begin{bmatrix}2x & 2y & -2z\\ 4x & +2y & -1\end{bmatrix} < 2$$

$\Leftrightarrow$ all 2×2 sub dets $= 0$

$$\Leftrightarrow \begin{cases} \det\begin{bmatrix}2x & 2y\\ 4x & 2y\end{bmatrix} = 0 \\ \det\begin{bmatrix}2y & -2z\\ 2y & -1\end{bmatrix} = 0 \\ \det\begin{bmatrix}2x & -2z\\ 4x & -1\end{bmatrix} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -\cancel{4}xy = 0 \\ -\cancel{2}y + \overset{2}{\cancel{4}}yz = 0 \\ -\cancel{2}x + \overset{4}{\cancel{8}}zx = 0 \end{cases} \Leftrightarrow \begin{cases} xy = 0 \\ y(-1+2z) = 0 \\ x(-1+4z) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y(2z-1) = 0 \\ \cancel{0 = 0} \end{cases} \vee \begin{cases} y = 0 \\ \cancel{0 = 0} \\ x(4z-1) = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ z = 1/2 \end{cases} \vee \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} y = 0 \\ z = 1/4 \end{cases}$$

$$\underset{z \in \mathbb{R}}{(0,0,z)} \qquad \underset{y \in \mathbb{R}}{(0,y,1/2)} \qquad \cancel{\underset{z \in \mathbb{R}}{(0,0,z)}} \qquad \underset{x \in \mathbb{R}}{(x,0,1/4)}$$

$$(0,0,z) \in \mathcal{M} \iff \begin{cases} z^2 = 1 \\ z = 0 \end{cases} \quad \text{imposs}$$

$$(0,y,1/2) \in \mathcal{M} \iff \begin{cases} \frac{1}{4} = y^2 + 1 \qquad y^2 = -\frac{3}{4} \quad \text{imposs} \\ \sim \end{cases}$$

$$(x,0,1/4) \in \mathcal{M} \iff \begin{cases} \frac{1}{16} = x^2 + 1 \Rightarrow x^2 = -\frac{15}{16} \\ \qquad\qquad\qquad \text{imposs.} \end{cases}$$

$$\Rightarrow \quad \operatorname{rk}[\vec{g}\,'] = 2 \quad \text{on } \mathcal{M}.$$

Q2. $\mathcal{M}$ is compact $\iff$ $\mathcal{M}$ is closed & bded

$$\mathcal{M} = \{g_1 = 0,\ g_2 = 0\}, \quad g_1, g_2 \in \mathcal{C}$$

$\Rightarrow$ m is closed.

Is m bdd ?

$$(x,y,z)\in m \iff \begin{cases} z^2 = x^2+y^2+1 \\ z = \underline{2x^2+y^2} = x^2 + (x^2+y^2) \end{cases}$$

$+$

$$z \geq \underline{x^2+y^2}$$

~~$x^2+y^2-z^2=-1$~~

~~$\Downarrow$~~

~~$x^2 \leq -1,\ y^2 \leq -1,\ z^2 \leq -1$~~

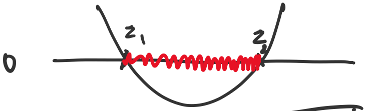
$$\Rightarrow \boxed{z \geq x^2+y^2}$$

$$z^2 = x^2+\underline{y^2+1} \leq z+1$$

$$\Downarrow$$

$$z^2 \leq z+1 \iff z^2-z-1 \leq 0$$

$$\Delta = 5$$



$$\frac{1-\sqrt{5}}{2} \leq \boxed{z \leq \frac{1+\sqrt{5}}{2}}$$

$$\Rightarrow x^2+y^2 \leq z \leq \frac{1+\sqrt{5}}{2} \Rightarrow x^2 \leq \frac{1+\sqrt{5}}{2}$$

$y^2 \leq \frac{1+\sqrt{5}}{2}$

$\Rightarrow \mathcal{M}$ is bdd $\Rightarrow \mathcal{M}$ is compact.

Q3. We have to det $\min_{\mathcal{M}}/\max f(x,y,z) \overset{\cdot\cdot}{=} z$

Existence: $f \in \mathcal{C}(\mathbb{R}^3) \Rightarrow f \in \mathcal{C}(\mathcal{M})$, $\mathcal{M}$ is compact

$\Rightarrow (W)$. $\exists$ both $\min_{\mathcal{M}}/\max f$.

Search: if $(x,y,z) \in \mathcal{M}$ is min/max pt for f, by Q1 and L. thm

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\Updownarrow$$

$$\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3$$

$$\Updownarrow$$

$$\operatorname{rk}\begin{bmatrix}0 & 0 & 1\\ 2x & 2y & -2z\\ 4x & 2y & -1\end{bmatrix} < 3$$

$$\Updownarrow$$

$$\det\begin{bmatrix} & // & \end{bmatrix} = 0$$

$$0 = 0\cdot\det\boxed{} - 0\cdot\det\boxed{} + 1\det\boxed{}$$

$$= 1\cdot\det\begin{bmatrix}2x & 2y\\ 4x & 2y\end{bmatrix}$$

$$= -4xy$$

So $\boxed{\operatorname{rk}\begin{bmatrix}\nabla f\\ \nabla g_1\\ \nabla g_2\end{bmatrix} < 3} \iff \boxed{xy = 0}$

$$\iff x = 0 \vee y = 0$$

$(0,y,z)$ $\quad$ $(x,0,z)$

$y,z\in\mathbb{R}$ $\quad$ $x,z\in\mathbb{R}$

$$(0,y,z) \in m \iff \begin{cases} z^2 = y^2 + 1 \\ z = y^2 \end{cases} \iff \begin{cases} z^2 = z + 1 \\ z = y^2 \end{cases}$$

$$\Rightarrow \boxed{\left(0, \pm\sqrt{\frac{1+\sqrt{5}}{2}}, \frac{1+\sqrt{5}}{2}\right)}$$

$$(x,0,z) \in m \iff \begin{cases} z^2 = x^2 + 1 \\ \frac{z}{2} = \cancel{2}x^2 \end{cases} \iff$$

$$\begin{cases} z^2 = \frac{z}{2} + 1 \\ x^2 = z/2 \end{cases}$$

$$\iff \begin{cases} 2z^2 - z - 2 = 0 \\ x^2 = z/2 \end{cases}$$

$$\iff \begin{cases} z = \frac{1 \pm \sqrt{17}}{4} \\ x^2 = z/2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} z = \dfrac{1+\sqrt{17}}{4} \\ x^2 = \dfrac{1+\sqrt{17}}{8} \end{cases}$$

$$\boxed{\left(\pm\sqrt{\frac{1+\sqrt{17}}{8}},\ 0,\ \frac{1+\sqrt{17}}{4} \right)}$$

Now, since

$$f\Big(\underbrace{0,\ \pm\sqrt{\cdots},\ \frac{1+\sqrt{5}}{2}}_{\text{max pts}}\Big) = \frac{1+\sqrt{5}}{2} \approx \frac{3}{2}$$

$$f\Big(\underbrace{\pm\sqrt{\cdots},\ 0,\ \frac{1+\sqrt{17}}{4}}_{\text{min pts}}\Big) = \frac{1+\sqrt{17}}{4} \approx \frac{5}{4}$$

■

3.6.20

$$\mathcal{M} = \{ x^2+y^2-z^2 = 0,\ \underbrace{x^2-y^2 = 1}\}$$

Q1 $\mathcal{M} \neq \emptyset$, $\mathcal{M} = \{\vec{g} = 0\}$ $\vec{g}$ subm. on $\mathcal{M}$

Q2 Is $\mathcal{M}$ compact

Q3 Det (if any) pts of $\mathcal{M}$ at min/max dist to $\vec{0}$.

Sol: $(x,y,z) \in M \iff \begin{cases} x^2 - y^2 = 1 \\ \underset{\curvearrowleft}{x^2} + y^2 - z^2 = 0 \end{cases}$

$$\begin{cases} x^2 = y^2 + 1 \\ z^2 = 2y^2 + 1 \end{cases}$$

$$\|(x,y,z)\|^2 =$$

$$= \underset{\uparrow}{x^2} + y^2 + z^2 =$$

$$= (y^2+1) + y^2 + 2y^2 + 1$$

$$= 4y^2 + 2 \xrightarrow[y \to +\infty]{} +\infty$$

$$\left(\pm\sqrt{y^2+1},\ y,\ \pm\sqrt{2y^2+1}\right) \in M$$

$$\|\qquad\qquad\qquad\qquad\| \xrightarrow[y \to \pm\infty]{} +\infty$$

$\Rightarrow$ $M \neq \emptyset$ and it is unbdd.

$$M = \{ g_1 = 0,\ g_2 = 0 \}$$

where $g_1 = x^2+y^2-z^2$

$g_2 = x^2-y^2-1$

(clearly g_1, g_2 are diff and so $\vec{g}=(g_1,g_2)$)

$\vec{g}=(g_1,g_2)$ is a submersion on $\mathcal{M}$

$$\iff \quad \operatorname{rk}\left[\vec{g}\,'\right] = 2 \text{ on } \mathcal{M}$$

$$\|$$

$$\operatorname{rk}\begin{bmatrix}\nabla g_1\\ \nabla g_2\end{bmatrix} = \operatorname{rk}\begin{bmatrix}2x & 2y & -2z\\ 2x & -2y & 0\end{bmatrix}$$

Now,

$$\operatorname{rk}\begin{bmatrix}2x & 2y & -2z\\ 2x & -2y & 0\end{bmatrix} < 2 \iff \text{all } 2\times2 \text{ subdets} = 0$$

$$\iff \begin{cases} \det\begin{bmatrix}2x & 2y\\ 2x & -2y\end{bmatrix} = 0\\ \det\begin{bmatrix}2x & -2z\\ 2x & 0\end{bmatrix} = 0\\ \det\begin{bmatrix}2y & -2z\\ -2y & 0\end{bmatrix} = 0\end{cases} \iff \begin{cases} -8xy = 0\\ 4xz = 0\\ -4yz = 0\end{cases}$$

$$\Leftrightarrow \begin{cases} xy = 0 \\ xz = 0 \\ yz = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ \cancel{0 = 0} \\ yz = 0 \end{cases} \quad \vee \quad \begin{cases} y = 0 \\ xz = 0 \\ \cancel{0 = 0} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 0 \\ z = 0 \end{cases} \vee \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} y = 0 \\ z = 0 \end{cases}$$

$$(0,0,z) \qquad (0,y,0) \qquad (x,0,0)$$

$$z \in \mathbb{R} \qquad y \in \mathbb{R} \qquad x \in \mathbb{R}$$

$$(0,0,z) \in M \Leftrightarrow \begin{cases} -z^2 = 0 \\ 0 = 1 \end{cases} \quad \text{imposs}$$

$$(0,y,0) \in M \Leftrightarrow \begin{cases} y^2 - z^2 = 0 \\ -y^2 = 1 \end{cases} \quad \text{imposs}$$

$$(x,0,0) \in M \Leftrightarrow \begin{cases} x^2 = 0 \\ x^2 = 1 \end{cases} \quad \text{imposs}$$

$\Rightarrow \quad \mathrm{rk}\,[\vec{g}\,'] = 2 \quad$ on $\quad \mathcal{M}$.

Q2. Is $\mathcal{M}$ compact?

By Q1, $\mathcal{M}$ is unbded ($\Rightarrow$ $\mathcal{M}$ is not compact)

$\mathcal{M} = \{g_1 = 0,\ g_2 = 0\}$, $\ g_1, g_2 \in \mathcal{C}$

$\Rightarrow$ $\mathcal{M}$ is closed

Q3 Since $\mathcal{M}$ is unbded, there are not pts of $\mathcal{M}$ at max dist to $\vec{0}$

About min dist, we have to minimize

$$f(x,y,z) = \sqrt{x^2+y^2+z^2}$$

or, equiv,

$$f(x,y,z) = x^2+y^2+z^2$$

Since

$$\lim_{(x,y,z)\to\infty_3} f = +\infty$$

and $f \in \mathcal{C}$, $\mathcal{M}$ is closed & unbded $\Rightarrow$

$\exists \min_{\mathcal{M}} f$. Let's find min pts applying

Lagr. thm. By Q1 $\vec{g}$ is a subm.

on $M = \{\vec{g} = 0\}$. Therefore, at $(x,y,z) \in M$ min pt for f

$$\boxed{\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2}$$

$$\iff \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3$$

$$\|$$

$$\operatorname{rk}\begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -2z \\ 2x & -2y & 0 \end{bmatrix} < 3$$

$$\iff \underset{x \cdot y\; z}{2 \cdot 2 \cdot 2} \cdot \det\begin{bmatrix} \cancel{2}\cancel{x} & \cancel{2}\cancel{y} & \cancel{2}\cancel{z} \\ \cancel{2}\cancel{x} & \cancel{2}\cancel{y} & -\cancel{2}\cancel{z} \\ \cancel{2}\cancel{x} & -\cancel{2}\cancel{y} & 0 \end{bmatrix} = 0$$

$$\iff \quad xyz \cdot \underset{\substack{\neq \\ 0}}{\det}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = 0$$

$$\iff \quad xyz = 0$$

$$\iff \quad x = 0 \quad \vee \quad y = 0 \quad \vee \quad z = 0$$

$$(0,y,z) \qquad (x,0,z) \qquad (x,y,0)$$

$y, z \in \mathbb{R}$ $\qquad$ $x, z \in \mathbb{R}$ $\qquad$ $x, y \in \mathbb{R}$.

$$(0, y, z) \in \mathcal{M} \iff \begin{cases} y^2 - z^2 = 0 \\ -y^2 = 1 \end{cases} \quad \text{imposs}$$

$$(x, 0, z) \in \mathcal{M} \iff \begin{cases} x^2 - z^2 = 0 \\ x^2 = 1 \end{cases} \implies \begin{cases} x = \pm 1 \\ z^2 = x^2 = 1 \end{cases}$$

$$2 \xleftarrow{f} (\pm 1, 0, \pm 1) \quad (4 \text{ pts})$$

$$(x, y, 0) \in \mathcal{M} \iff \begin{cases} x^2 + y^2 = 0 \\ x^2 - y^2 = 1 \end{cases} \quad \text{imposs}$$

Concl: $(\pm 1, 0, \pm 1)$ are pts of $\mathcal{M}$ at min dist to $\vec{0}$. ∎

Do 3.6.19/21/22/23/25/27/28