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Def: Let $g = g(x,y)$. We say that g is a submersion on D if

$$\nabla g \neq \vec{0} \qquad \forall (x,y) \in D$$

We can recast L. thm under the following form:

Thm: Assume that g is a submersion on $D = \{g = 0\}$.
Then, at min/max pt (x^*, y^*) for f on D we have

$$\boxed{\exists \lambda \in \mathbb{R} : \nabla f(x^*, y^*) = \lambda \nabla g(x^*, y^*)}$$

Example: min/max $(x+y)$

$$x^2 + 4y^2 = 1$$

Sol:

Existence: $f = x + y \in \underline{\mathcal{C}(D)}$, $D = \{x^2 + 4y^2 = 1\}$

$\underline{D}$ is clearly closed & bdd

$\Rightarrow (W) \quad \exists \ \underset{D}{\min/\max} f .$

Search : To det min/max pts, we apply L. thm.

Let's first verify that g is a submersion on $\{g=0\}$ $\quad g = x^2+4y^2-1$

$\Leftrightarrow \nabla g \neq \vec{0} \quad \forall (x,y) \in D = \{g=0\}.$

Now

$$\nabla g = \vec{0} \Leftrightarrow (2x, 8y) = \vec{0}$$

$$\Uparrow$$

$$\left.\begin{array}{l} \partial_x g = 2x \\ \partial_y g = 8y \end{array}\right. \in \mathcal{C}(\mathbb{R}^2) \Rightarrow g \text{ diff}$$

$$\Leftrightarrow (x,y) = \vec{0} \notin D$$

$\Rightarrow \nabla g \neq \vec{0}$ on D.

$\Rightarrow g$ is a submersion on D

Then at (x,y) min/max pts for f on $\underline{\underline{D}}$

$$\exists \lambda : \boxed{\nabla f(x,y) = \lambda \nabla g(x,y)} \quad (*)$$

$$\nabla f = (1,1)$$

$$\left.\begin{array}{l} \partial_x f = 1 \\ \partial_y f = 1 \end{array}\right. \in \mathcal{C} \Rightarrow f \text{ is diff}$$

$$(*) \iff (1,1) = \lambda(2x, 8y)$$

$$\iff \begin{cases} 1 = 2\lambda x \\ 1 = 8\lambda y \end{cases}$$

$\lambda \neq 0$ because $\lambda = 0 \Longrightarrow \begin{cases} 1 = 0 \\ 1 = 0 \end{cases}$ no sols

$$\iff \begin{cases} x = 1/2\lambda \\ y = 1/8\lambda \end{cases}$$

$$(x,y) = \left(\frac{1}{2\lambda}, \frac{1}{8\lambda}\right) \in D \ ?$$

$$\Updownarrow$$

$$\left(\frac{1}{2\lambda}\right)^2 + 4\left(\frac{1}{8\lambda}\right)^2 = 1$$

$$\iff \frac{1}{4\lambda^2} + \cancel{4}\,\frac{1}{\underset{16}{\cancel{64}}\lambda^2} = 1$$

$$\iff 16\lambda^2 = 5 \qquad \lambda^2 = 5/16$$

$$\Longrightarrow \lambda = \pm\frac{\sqrt{5}}{4}$$

$$\Longrightarrow (x,y) = \left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right), \left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}}\right)$$

$$\downarrow f \qquad\qquad\qquad \downarrow$$

$$\frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}} \qquad - \left(\qquad \right)$$

$\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right)$ is max pt for f on D

$\left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}}\right)$ is min 〃 〃 〃 〃 〃.

■

Rmk: Cond

$$\nabla f = \lambda \nabla g \qquad \text{lines } 2\left\{ \overbrace{\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix}}^{2 \text{ cols.}} \right.$$

the two lines are lin dep.

$$\Updownarrow$$

$$\boxed{\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0}$$

In prev example this cond writes

$$\det \begin{bmatrix} 1 & 1 \\ 2x & 8y \end{bmatrix} = 0$$

$$\boxed{8y - 2x = 0}$$

$$\Leftrightarrow \quad y = \frac{x}{4} \quad \Rightarrow \quad \left(x, \frac{x}{4}\right) \; x \in \mathbb{R}$$

These are all sols of $\nabla f = \lambda \nabla g$

$$(x, x/4) \in D \iff x^2 + 4\left(\frac{x}{4}\right)^2 = 1$$

$$\iff x^2 + 4\,\frac{x^2}{16} = 1$$

$$\iff 5x^2 = 4$$

$$\iff x = \pm\frac{2}{\sqrt{5}}$$

$$\Rightarrow \left(\frac{2}{\sqrt{5}}, \frac{2}{4\sqrt{5}}\right), \quad \left(-\frac{2}{\sqrt{5}}, -\frac{1}{2\sqrt{5}}\right)$$

$$\left(\frac{2}{\sqrt{5}}, \frac{1}{2\sqrt{5}}\right)$$

$\blacksquare$

Do exs 3.6.15

iii) min/max $f(x,y) = xy$

$$\underbrace{x^2+y^2+xy-1}=0$$

Sol: Existence: $f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow f \in \mathcal{C}(D)$

$D = \{g = 0\}$, $g \in \mathcal{C} \Rightarrow D$ is closed.

Is D also bded?

$$x^2+y^2+xy-1=0$$

$$\underset{+}{x^2}+\underset{+}{y^2}+\underbrace{xy}_{?} = 1$$

$$x = \rho\cos\theta$$

$$y = \rho \sin\theta$$

$$\Leftrightarrow \quad \rho^2 + \rho^2 \underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta} = 1$$

$$\Leftrightarrow \quad \rho^2 \Big(\underbrace{1 + \tfrac{1}{2}\sin 2\theta}_{+} \Big) = 1$$

$$\overset{?}{\Rightarrow} \quad \rho^2 \leqslant C \quad ?$$

$$\Leftrightarrow \quad \rho^2 = \frac{1}{1 + \frac{1}{2}\sin(2\theta)} \leqslant C$$

$$1/2 = 1 - 1/2 \leqslant \qquad\qquad \leqslant 1 + 1/2 = 3/2$$

$$\rho^2 \leqslant \frac{1}{1/2} = 2$$

$$\Rightarrow \quad \rho \leqslant \sqrt{2} \quad \Leftrightarrow \quad \|(x,y)\| \leqslant \sqrt{2}.$$

$\Rightarrow$ D is bdd.

$\Rightarrow$ by W. thm $\exists \underset{D}{\min/\max} f$.

Search: We wish to apply L. thm.

To this aim let's first verify g submersion on D $\Leftrightarrow$ $\nabla g \neq \vec{0}$ on $D = \{g = 0\}$.

Now $g = x^2 + y^2 + xy - 1$

$$\nabla g = (2x + y,\ 2y + x) = \vec{0}$$

$$\left.\begin{aligned} \partial_x g &= 2x + y \\ \partial_y g &= 2y + x \end{aligned}\right\} \in \mathscr{C}(\mathbb{R}^2) \Rightarrow g \text{ is diff on } \mathbb{R}^2$$

$$\Leftrightarrow \quad \begin{cases} 2x + y = 0 \\ 2y + x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

however, since $(0,0) \in D \Leftrightarrow -1 = 0$ false

$$\Rightarrow \nabla g \neq \vec{0} \quad \forall\, (x,y) \in D.$$

Then, if $\boxed{(x,y) \in D}$ is min/max for f on D $\Rightarrow \boxed{\nabla f = \lambda \nabla g}$

$$\Leftrightarrow \quad \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \det \begin{bmatrix} y & x \\ 2x + y & 2y + x \end{bmatrix} = 0$$

$$\Leftrightarrow \quad y(2y + \cancel{x}) - x(2x + \cancel{y}) = 0$$

$$\Leftrightarrow \quad \cancel{2}y^2 - \cancel{2}x^2 = 0$$

$$\Leftrightarrow \quad (y-x)(y+x)=0 \quad \Leftrightarrow$$

$$y = x \qquad \vee \qquad y = -x$$

$$\Updownarrow$$

$$(x,x) \qquad \vee \qquad (x,-x)$$

$$x \in \mathbb{R} \qquad\qquad x \in \mathbb{R}$$

$$(x,x) \in D \quad \Leftrightarrow \quad x^2 + x^2 + x^2 - 1 = 0$$

$$\Leftrightarrow \quad 3x^2 = 1, \quad x^2 = 1/3$$

$$\Leftrightarrow \quad x = \pm \frac{1}{\sqrt{3}}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \qquad \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

$$\swarrow f \qquad\qquad \swarrow f$$

$$1/3 \qquad\qquad 1/3$$

$$(x,-x) \in D \quad \Leftrightarrow \quad x^2 + (-x)^2 + x \cdot (-x) - 1 = 0$$

$$x^2 + \cancel{x^2} - \cancel{x^2} - 1 = 0$$

$$\Leftrightarrow \quad x^2 = 1 \quad \Leftrightarrow \quad x = \pm 1$$

$$(1,-1), \qquad (-1,1).$$

$$\swarrow f \qquad\qquad \swarrow f$$

$$-1 \qquad\qquad -1$$

Conclusion: $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are max pts for f on D

$(1,-1)$, $(-1,1)$ are min pts for f on D. ■

Extensions of Lagrange Thm

So far, we discussed the pb

$$\text{min/max } f(x,y)$$
$$g(x,y)=0$$

What if we have

$$\text{min/max } f(\vec{x})$$
$$g(\vec{x})=0$$

where
$$f=f(\vec{x}) : S \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$
$$g=g(\vec{x}) : S \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

Def: We say that g is a submersion on D if

$$\nabla g \neq \vec{0} \quad \forall\ \vec{x} \in D.$$

Thm: (Lagrange)

Let g be a submersion on $D=\{g=0\}$.
If $\vec{x}^*$ is a min/max pt for f (differentiable) on $D=\{g=0\}$, then

$$\nabla f(\vec{x}^*) = \lambda \nabla g(\vec{x}^*).$$

Rmk Cond

$$\nabla f = \lambda \nabla g \iff 2\overbrace{\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix}}^{d} \text{ has the two lines lin. dep}$$

$$2\times d$$

$$\iff \operatorname{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} = 1 \quad (\text{because } g \text{ is a subm.})$$

$$\iff \operatorname{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} < 2$$

Now

$$\operatorname{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} < 2$$

$$\begin{bmatrix} \partial_1 f & \partial_2 f & \partial_3 f & \cdots & \partial_d f \\ \partial_1 g & \partial_2 g & \partial_3 g & \cdots & \partial_d g \end{bmatrix}$$

$$\begin{bmatrix} \partial_x f & \partial_y f & \partial_z f \\ \partial_x g & \partial_y g & \partial_z g \end{bmatrix}$$

$$\ulcorner \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \iff \det\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0$$

IS WRONG unless $d=2$ $\lrcorner$

$$\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \iff \text{all } 2\times 2 \text{ subdeterminants vanish}$$

Ex. 3.6.16

i) $\min/\max_{D} f$ $\qquad f = x - 2y + 2z$

$D = \{x^2+y^2+z^2 = 9\}$

$g = x^2+y^2+z^2-9 = 0$

Sol: Existence: $f \in \mathcal{C}(\mathbb{R}^3) \Rightarrow f \in \mathcal{C}(D)$

D is clearly closed & bded

$\Downarrow$

$$(W.)\quad \exists \underset{D}{\min / \max} f .$$

Search : we apply the L. thm.

Let's start verifying that g is a submersion on D

$\nabla g = \vec{0}$

$(\partial_x g = 2x,\ \partial_y g = 2y,\ \partial_z g = 2z,$

$\in \mathcal{C}(\mathbb{R}^3) \Rightarrow g$ is diff on D)

$\Leftrightarrow \quad (2x, 2y, 2z) = (0,0,0)$

$\Leftrightarrow \quad (x,y,z) = (0,0,0) \notin D \quad \Rightarrow$

$\nabla g \neq \vec{0}$ on D.

At $\boxed{(x,y,z) \in D}$ min / max pt for f on D

$$\boxed{\nabla f = \lambda \nabla g}$$

$$\Updownarrow$$

$$rk \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2$$

$$\Updownarrow \qquad f = x - 2y + 2z$$

$$\operatorname{rk}\begin{bmatrix} 1 & -2 & +2 \\ 2x & 2y & 2z \end{bmatrix} < 2$$

$$\Leftrightarrow \begin{cases} \det\begin{bmatrix} 1 & -2 \\ 2x & 2y \end{bmatrix} = 0 \\ \det\begin{bmatrix} -2 & 2 \\ 2y & 2z \end{bmatrix} = 0 \\ \det\begin{bmatrix} 1 & 2 \\ 2x & 2z \end{bmatrix} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \cancel{2}y + \overset{2}{\cancel{4}}x = 0 \\ +\cancel{4}z + \cancel{4}y = 0 \\ \cancel{2}z - \overset{2}{\cancel{4}}x = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y + 2x = 0 \\ z + y = 0 \\ z - 2x = 0 \end{cases} \qquad \begin{cases} y = -2x \\ \\ z = 2x \end{cases}$$

$$\Rightarrow \quad (x, -2x, 2x) \qquad x \in \mathbb{R}$$

$$(x, -2x, 2x) \in D \quad \Leftrightarrow \quad x^2 + (-2x)^2 + (2x)^2 = 9$$

$$9x^2 = 9$$

$$\iff \quad x^2 = 1 \iff x = \pm 1$$

$$\overset{\text{max pt}}{(1,-2,2)} \xrightarrow{f} 1+4+4 = 9$$

$$\underset{\text{min pt.}}{(-1,2,-2)} \xrightarrow[f]{} -1-4-4 = -9$$

Do 3.6.16 ii)

The next (and last) extension is to the pb

$$\min/\max \quad f(\vec{x})$$

$$\begin{cases} g_1(\vec{x}) = 0 \\ g_2(\vec{x}) = 0 \\ \vdots \\ g_m(\vec{x}) = 0 \end{cases}$$

Def: We say that $\vec{g} = \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = \left(g_1(\vec{x}), g_2(\vec{x}), --- \vec{g}_m(\vec{x})\right)$$

is a submersion on D if

$$\operatorname{rk}\left[\vec{g}\,'(\vec{x})\right] = \operatorname{rk}\begin{bmatrix} \nabla g_1(\vec{x}) \\ \nabla g_2(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = m.$$

Rmk: To check this cond we usually look for pts where

$$\operatorname{rk}\left[\vec{g}\,'(\vec{x})\right] < m \iff \text{all } m\times m \text{ subdeterminant} = 0$$

Thm: Let $\vec{g} = (g_1, \dots, g_m)$ be a submersion on $D = \{g_1 = 0, g_2 = 0, \dots, g_m = 0\}$. If $\vec{x}^*$ is min / max for f on D then

$$(L)\quad \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \dots + \lambda_m \nabla g_m.$$

Rmk:

$$(L) \iff \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \vdots \\ \nabla g_m \end{bmatrix} = m$$

$\iff$ all $m+1 \times m+1$ sub dets vanish.