Thursday, 30 November 2023 08:46

Constrained Optimization

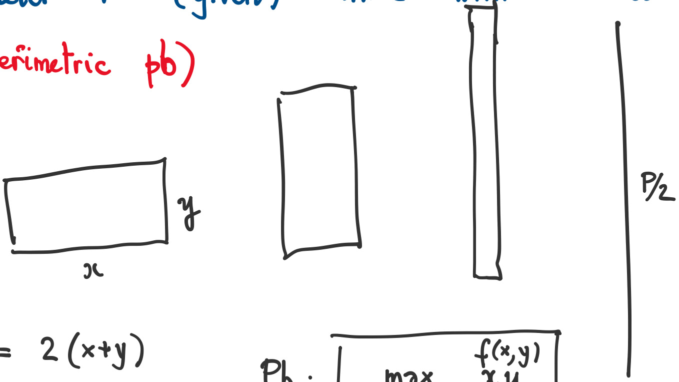
We consider now

$$\min/\max_{\vec{x} \in D} f(\vec{x})$$

$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ when $\text{Int } D = \emptyset$.

Pb: Det., among all rectangles with perimeter P (given) those with max area

(isoperimetric pb)

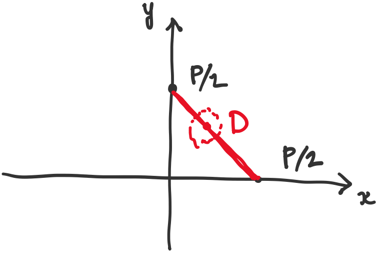


$P = 2(x+y)$

$A = x \cdot y$

Pb: $$\boxed{\begin{array}{c} \max\limits_{x,y} \overset{f(x,y)}{xy} \\ 2(x+y) = P,\ x \geq 0,\ y \geq 0 \end{array}}$$

$$\underbrace{\qquad\qquad\qquad}_{D}$$



$$D = \{(x,y) : \overset{x \geqslant 0,\ y \geqslant 0}{2(x+y) = P}\}$$

$$\Updownarrow$$

$$x+y = P/2$$

$$y = \frac{P}{2} - x$$

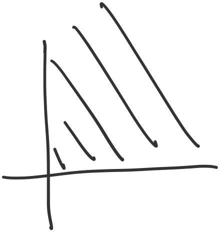
$$\operatorname{Int} D = \emptyset .$$

$$\Downarrow$$

$\nabla f = 0$ is not nec. verified at min/max

How can we solve

$$\max \boxed{xy} \quad \underbrace{\boxed{2(x+y) = P}, \boxed{x \geqslant 0,\ y \geqslant 0}}_{D}$$



Existence: $f(x,y) = xy \in \mathcal{C}(\mathbb{R}^2) \Rightarrow f \in \mathcal{C}(D)$

D is closed, D bded

$\Rightarrow D$ is compact

$\Rightarrow$ (W) $\exists \min_{D}/\max f$.

Search: (for min $x = 0 \quad y = P/2$; $x = P/2 \quad y = 0$ are min pts)

$$(x,y) \in D \iff \boxed{x \geq 0} \ \boxed{y \geq 0} \quad x + y = P/2$$

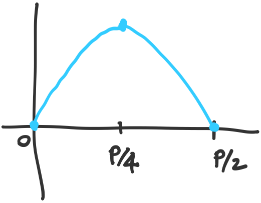
$$\Updownarrow$$

$$0 \leq y = P/2 - x$$

$$\boxed{f(x,y)} = f\left(\overbracket{x, \frac{P}{2} - x}\right)$$

$$= x\left(\frac{P}{2} - x\right) = \frac{P}{2} x - x^2 \ \boxed{= h(x)}$$

f is max on D $\iff$ h is max on $x \in [0, P/2]$



$$h(0) = 0$$

$$h(P/2) = 0$$

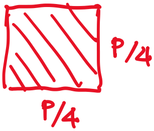
$$h' = \frac{P}{2} - 2x \geq 0$$

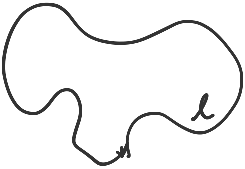
$$\iff \quad x \leq P/4$$

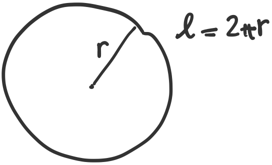
Since h is max at $x = P/4$

$$\Downarrow$$

f is max at $(P/4, P/4)$







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In general, let's consider

$$\underset{g(x,y)=0}{\min/\max} \; f(x,y)$$

$$= \underset{D}{\min/\max} \; f(x,y)$$

where $D = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$.

Examples of domains def'd in this way.

- D can be empty

$$D = \{x^2+y^2+1=0\} = \phi$$

- D can be a singleton

$$D = \{x^2+y^2=0\} = \{(0,0)\}$$

- D can be everything

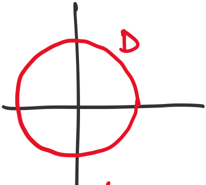
$$D = \{0=0\} = \mathbb{R}^2$$

- but, in general, a set

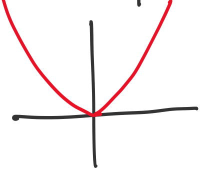
$$D = \{g(x,y)=0\}$$

is a line in $\mathbb{R}^2$

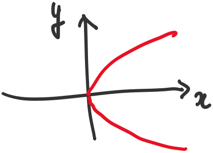
$$D = \{x^2+y^2=1\}$$



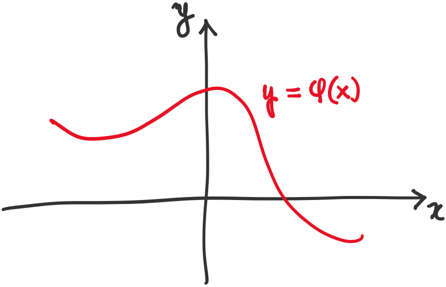
$$D = \{y = x^2\}$$

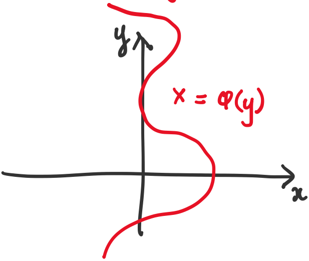


$D = \{x = y^2\}$

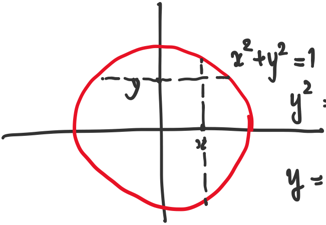


Idea : $g(x,y) = 0 \Longleftrightarrow y = \varphi(x)$ or $x = \varphi(y)$





$D = \{x^2 + y^2 = 1\}$



$x^2+y^2=1$

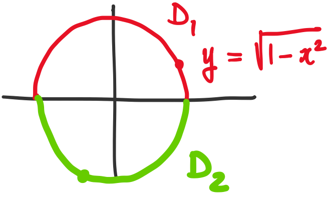
$y^2 = 1-x^2$

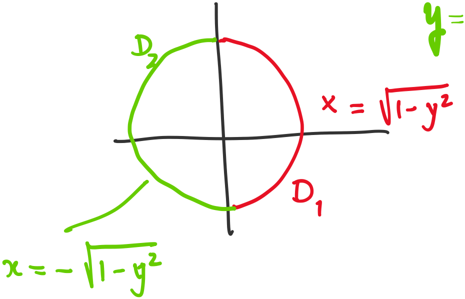
$y = \pm\sqrt{1-x^2}$

This D is not of type

$y = \varphi(x)$ or $x = \varphi(y)$.

but $D = D_1 \cup D_2$





So, if

$$D = \{\boxed{g(x,y) = 0}\} = \{\boxed{y = \varphi(x)}\}$$

$$\max_{D}/\min_{D} f(x,y) = \max_{I}/\min_{I} f(x, \varphi(x))$$

The pb is that, in general, it might be impossible to solve

$$g(x,y)=0 \quad \text{to obtain} \quad y=\varphi(x)$$

or

$$x=\varphi(y).$$

Pb: $\min/\max f(x,y)$

$g(x,y)=0$

Goal: Find a cond to be verified at min/max pts

Assume (for now)

$$g(x,y)=0 \iff \overset{x=\varphi(y)}{y=\varphi(x)}.$$

$$\Rightarrow \underset{\substack{g(x,y)=0 \\ (x^*,y^*)}}{\min/\max} f(x,y) = \underset{\substack{x\in I \\ x^*}}{\min/\max} \underset{\substack{\| \\ h(x)}}{\boxed{f(x,\varphi(x))}}$$

$$\Rightarrow \qquad h'(x_{min}) = 0$$

$$h'(x_{max}) = 0$$

$$h'(x) = \frac{d}{dx}\left[f(x, \varphi(x))\right]$$

Tot. deriv.

$$\frac{d}{dt} f(\vec{\gamma}(t)) = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}\,'(t)$$

In our case

$$\vec{\gamma}(x) = (\overset{\downarrow}{x}, \overset{\downarrow}{\varphi(x)}) \qquad \vec{\gamma}\,'(x) = \overset{\partial_x}{\begin{bmatrix} 1 \\ \varphi'(x) \end{bmatrix}}$$

$$\Rightarrow h'(x) = \frac{d}{dx} f(\vec{\gamma}(x)) = \nabla f(\vec{\gamma}(x)) \cdot \vec{\gamma}\,'(x)$$

$$= \nabla f(x, \varphi(x)) \cdot (1, \varphi'(x))$$

$$= \partial_x f \quad 1 \quad + \quad \partial_y f \quad \varphi'(x)$$

$$\underbrace{h'(x)} = \underbrace{\partial_x f(x, \varphi(x)) + \partial_y f(x, \varphi(x))\, \varphi'(x)}$$

So if x^* is min/max pt for h

$$h'(x^*) = 0$$

$$\Downarrow$$

$$(1) \quad \boxed{0 = \partial_x f(x^*, y^*) + \partial_y f(x^*, y^*) \, \overset{\downarrow}{\varphi'}(x^*)}$$

Second Step: let's compute $\varphi'(x)$

$$g(x,y) = 0 \iff y = \varphi(x)$$

$$\Rightarrow \qquad g(x, \varphi(x)) \equiv 0$$

$$\boxed{\frac{d}{dx} g(x, \varphi(x))} = 0$$

$$0 = \underbrace{\partial_x g(x, \varphi(x))}_{\curvearrowleft} + \partial_y g(x, \varphi(x)) \, \varphi'(x)$$

$$\Rightarrow \qquad \underline{\varphi'(x) = - \frac{\partial_x g(x, \varphi(x))}{\partial_y g(x, \varphi(x))}}$$

$$\Rightarrow \quad \boxed{\varphi'(x^*) = - \frac{\partial_x g(x^*, y^*)}{\partial_y g(x^*, y^*)}}$$

provided $\partial_y g(x^*, y^*) \neq 0$

$\partial_x g(\qquad) \neq 0$

Plugging this formula into (1) we get

$$0 = \partial_x f + \partial_y f \cdot \left(- \frac{\partial_x g}{\partial_y g} \right)$$

$$\Updownarrow$$

$$\partial_x f \, \partial_y g - \partial_y f \, \partial_x g = 0$$

$$\partial_x f \, \partial_y g + \partial_y f \, (-\partial_x g) = 0$$

$$(\partial_x f, \partial_y f) \cdot (\partial_y g, -\partial_x g) = 0$$

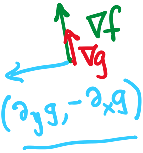
$$\nabla f \cdot (\qquad\qquad) = 0$$

$$\Updownarrow$$

$$\nabla f \perp (\partial_y g, -\partial_x g).$$

(a,b)

$(-b,a)$



$$\perp \quad (+\partial_x g, \partial_y g) = \nabla g$$

$$\Updownarrow$$

$$\nabla f \parallel \nabla g$$

$$\Updownarrow$$

$$\exists \lambda \in \mathbb{R} : \nabla f = \lambda \nabla g$$

Thm: If (x^*, y^*) is min/max pt for f on $D = \{g = 0\}$, and

$$\partial_y g(x^*, y^*) \neq 0$$

$$\Rightarrow \exists \lambda \in \mathbb{R} : \nabla f(x^*, y^*) = \lambda \nabla g(x^*, y^*).$$

By repeating the same "informal proof"

done above assuming $g(x,y)=0 \Leftrightarrow x=\varphi(y)$

we obtain a similar statement

Thm: (x^*, y^*) is min/max pt for f on $D=\{g=0\}$ and

$$\partial_x g(x^*,y^*) \neq 0$$

$$\Rightarrow \qquad \nabla f(x^*,y^*) = \lambda \nabla g(x^*,y^*)$$

Merging the two statements we get

Thm (Lagrange)

If (x^*, y^*) is min/max pt for f on $D=\{g=0\}$ and

$$\boxed{\nabla g(x^*,y^*) \neq \vec{0}}$$

$$\Rightarrow \qquad \nabla f(x^*,y^*) = \lambda \nabla g(x^*,y^*).$$

Example

$$\text{min/max} \quad xy$$

$$x^2+y^2=1$$

Sol: $f(x,y) = xy$

$D = \{x^2+y^2=1\} = \{g=0\}$

$g(x,y) = x^2+y^2-1$

Existence: D is compact, $f \in \mathcal{C}(D)$
$\Rightarrow$ (W) $\exists$ min/max for f on D

Search: We apply L. thm.
We verify if $\nabla g = \vec{0}$?

$$\nabla g = (\partial_x g, \partial_y g) = (2x, 2y)$$

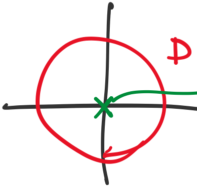
$$(\partial_x g, \partial_y g \in \mathcal{C} \Rightarrow g \text{ diff})$$

$$\nabla g = \vec{0} \iff (2x, 2y) = \vec{0}$$

$$(\Rightarrow) \quad (x,y) = \vec{0} \notin D$$

$$\Downarrow$$

$$\nabla g \neq \vec{0} \text{ on } D$$



So, if $(x,y) \in D$ is min/max pt for f

$$\nabla f = \lambda \nabla g = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix} \quad (*)$$

$$\nabla f = (\partial_x f, \partial_y f) = (y, x)$$

$$(\partial_x f, \partial_y f \in \mathcal{C} \Rightarrow f \text{ is diff on } D)$$

$$(*) \quad \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\begin{cases} \boxed{y = 2\lambda x} \\ x = 2\lambda y \end{cases}$$

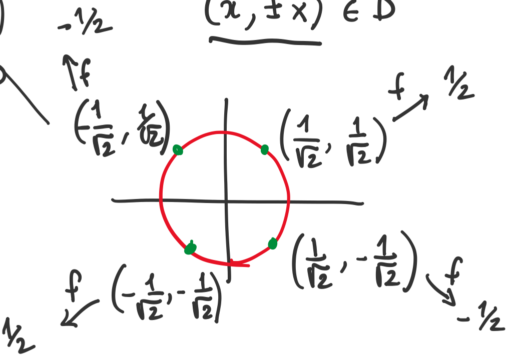
$$\Leftrightarrow \begin{cases} y = 2\lambda x \\ x = 2\lambda \cdot 2\lambda x = 4\lambda^2 x \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 2\lambda x \\ x(1-4\lambda^2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} \lambda^2 = 1/4 \quad \lambda = \pm 1/2 \\ y = \pm x \end{cases}$$

$\xcancel{(0,0) \notin D}$

$\underline{(x, \pm x)} \in D$



$-1/2$
$\uparrow f$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$f \nearrow 1/2$
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$f \searrow -1/2$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$f \swarrow 1/2$

$\pm \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are max pts

$\pm \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are min pts.