Wednesday, 29 November 2023 08:37

Ex 3.6.6.

$$f(x,y) := x^2(1-y)$$

$$D = \{(x,y) \in \mathbb{R}^2 : \quad x^2 + |y| \leq 4\}$$

i) sgn f

ii) st pts for f on D

iii) min/max f on D

iv) $f(D)$.

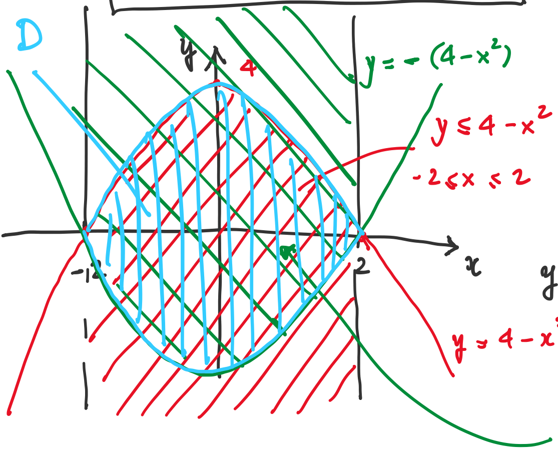
Sol: We start drawing D.

$$(x,y) \in D \iff x^2 + |y| \leq 4$$

$$\iff |y| \leq \underbrace{4 - x^2}$$

$$\overbrace{\overbrace{-(4-x^2) \leq y} \leq \underbrace{4-x^2}}$$

$x: \; 4 - x^2 \geq 0$



$$\Updownarrow$$

$$x^2 \leq 4$$

$$\Updownarrow$$

$$\boxed{-2 \leq x \leq 2}$$

$$y = 4-x^2$$

i) sgn f :

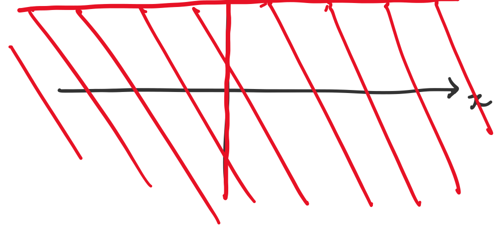
$$f(x,y) \geq 0 \iff x^2(1-y) \geq 0$$

$$\iff \boxed{x = 0 \qquad 0 \geq 0 \quad \forall y} \quad (0,y)\ \forall y$$

or

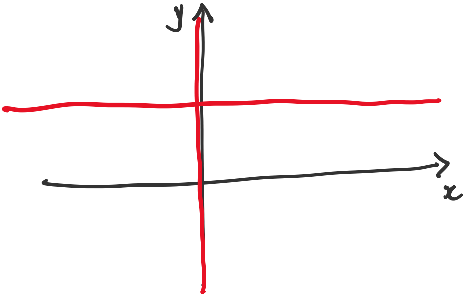
$$x \neq 0 \iff 1-y \geq 0$$

$$y \leq 1$$



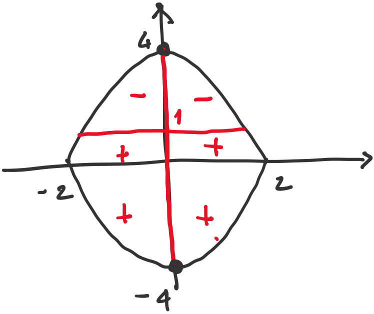
$f \geq 0$

$$f = 0 \iff x^2(1-y) = 0 \iff x^2 = 0 \vee \underset{= 0}{1-y}$$



$$\iff \underset{\forall y}{x=0} \vee \underset{\forall x}{y=1}$$

$f \geq 0$ in $D \iff$



ii) Diff : we apply diff test

$$\left.\begin{array}{l} \partial_x f = 2x(1-y) \\ \partial_y f = -x^2 \end{array}\right\} \in \mathcal{C}(\mathbb{R}^2)$$

$\Rightarrow$ f is diff on $\mathbb{R}^2$ ($\Rightarrow$ on D)

$(x,y) \in D$ is st. pt for f

$$\iff \nabla f = \vec{0} \iff \begin{cases} 2x(1-y) = 0 \\ -x^2 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ \cancel{0 = 0} \end{cases} \iff \boxed{\begin{array}{c} (0,y) \\ -4 \leq y \leq 4 . \end{array}}$$

iii) min/max for f on D.

Existence: D is clearly closed (because D is defd by a large ineq for a cont funct) and bdd

(see the figure)

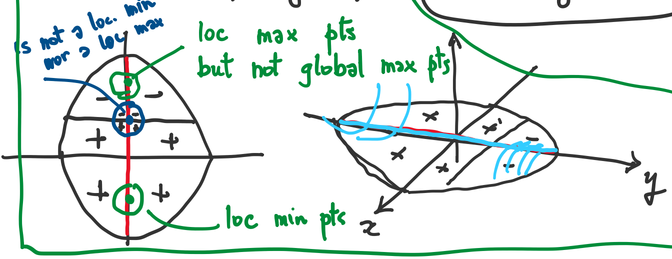
$\Rightarrow \exists$ global min pt
〃 max pt
for f on D.

Search: If $(x,y) \in D$ is min/max pt then

- either $(x,y) \in \operatorname{Int} D \Rightarrow$ (Fermat)

$\Rightarrow \nabla f(x,y) = \vec{0} \iff (x,y) = \boxed{(0,y)\quad -4<y<4}$

Notice that, by i)


is not a loc. min nor a loc max
loc max pts but not global max pts
loc min pts
−
−
+
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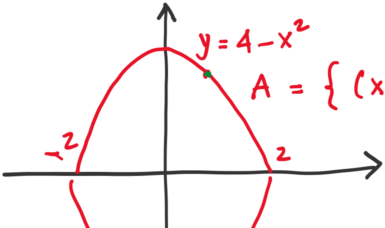
y
z

- or $(x,y) \in D \setminus \operatorname{Int} D = \partial D$



$A = \{ (x,y) : \underline{y = 4-x^2},\ -2 \leq x \leq 2 \}$

$= \{ (x, \underline{4-x^2}) : -2 \leq x \leq 2 \}$

$B = \{(x, x^2-4) : -2 \leq x \leq 2\}$

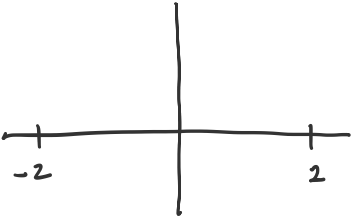
$y = -(4-x^2)$

$= x^2 - 4$

On A $\quad f(x,y) = x^2(1-y)$

$f \overset{(x,4-x^2)}{=} \boxed{f(x,4-x^2)} = x^2(1-(4-x^2))$

$= x^2(-3+x^2) =: \boxed{h(x)}$



$\boxed{\text{Pb: det min/max for } h \text{ when } x \in [-2,2]}$

$\Updownarrow$

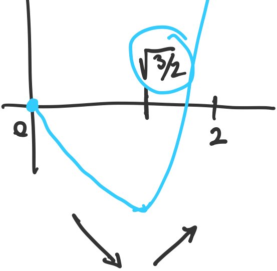
det min/max for f when $(x,y) \in A$

Since h depends on x^2 we can limit the search to $[0,2]$

$h(x) = x^2(x^2-3)$

$h(0) = 0$

$h(2) = 4 \cdot 1 = 4$



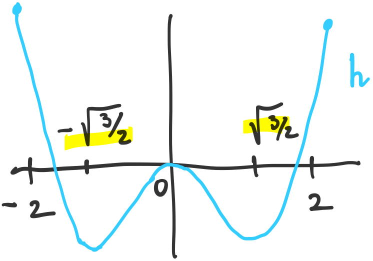
$$h' = 2x(x^2-3) + x^2 2x$$

$$= 2x(x^2 - 3 + x^2)$$

$$= \underbrace{2x}_{}(2x^2 - 3) \geqslant 0 \quad \overset{0 \leqslant x \leqslant 2}{\iff} \quad 2x^2 - 3 \geqslant 0$$

$$\iff \quad x^2 \geqslant 3/2$$

$$\overset{0 \leqslant x \leqslant 2}{\iff} \quad x \geqslant \sqrt{3/2}$$



min pts for h are $x = \pm\sqrt{3/2}$

max " " " " $x = \pm 2$

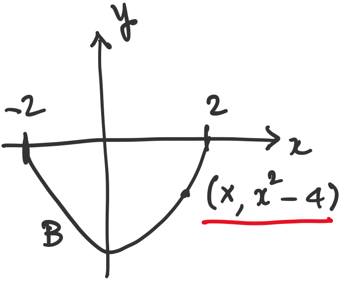
$\Downarrow$

min pts for f on A are $\left(\pm\sqrt{3/2},\ 4 - \frac{3}{2}\right)$

$$\boxed{= \left(\pm\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right)}$$

max pts for f on A are $\boxed{(\pm 2, 0)} \leftarrow$

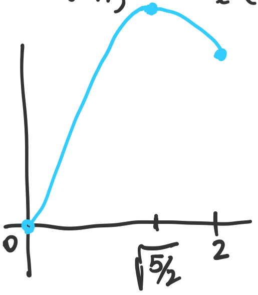
We have to do the same job on B:



$$f(x, x^2-4)$$
$$= x^2(1-(x^2-4))$$
$$= x^2(5-x^2) =: h(x)$$

min/max pts for f on $B \Longleftrightarrow$ min/max pts for h on $x \in [-2,2]$

h even, $x \in [0,2]$



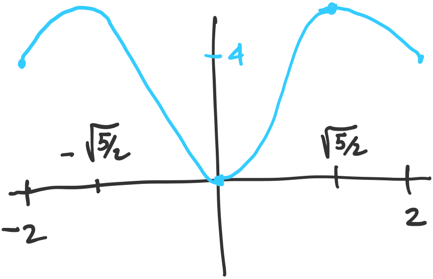
$$h(0) = 0,$$
$$h(2) = 4 \cdot 1 = 4$$
$$h' = 2x(5-x^2) + x^2(-2x)$$
$$= 2x(5-2x^2) \geq 0$$

↗ ↘

$$\overset{x\in[0,2]}{\iff} \quad 5-2x^2 \geqslant 0$$

$$\iff \quad x^2 \leqslant 5/2$$

$$\overset{x\in[0,2]}{\iff} \quad 0 \leqslant x \leqslant \sqrt{5/2}$$



h has min pts at $x = 0$

h " max pts at $x = \pm\sqrt{5/2}$

$\Downarrow$

f has min pt on B at $(0,-4)$

" " max " " " " $\left(\pm\sqrt{5/2}, \frac{5}{2}-4\right)$

$$= \left(\pm\sqrt{\frac{5}{2}}, -3/2\right)$$

Conclusion: If $(x,y)\in D$ is min $\Rightarrow$

(x,y) can be only one of these pts

$$(0,\underline{\underline{y}}) \quad -4<y<4 \quad \xrightarrow{f} \quad 0$$

$$\boxed{\left(\pm\sqrt{\frac{3}{2}}, \frac{5}{2}\right)} \quad \longrightarrow \quad \frac{3}{2}\cdot\left(1-\frac{5}{2}\right) = \boxed{-\frac{9}{4}}$$

$$(0,-4) \quad \longrightarrow \quad 0$$

$\Rightarrow \left(\pm\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ are min pts for f on D

For max the poss. pts are

$$(0,y) \quad -4<y<4 \quad \xrightarrow{f} \quad 0$$

$$(\pm 2, 0) \quad \longrightarrow \quad 4$$

$$\left(\pm\sqrt{\frac{5}{2}}, -3/2\right) \quad \longrightarrow \quad \frac{5}{2}\cdot\frac{5}{2} = \boxed{\frac{25}{4}}$$

$\left(\pm\sqrt{\frac{5}{2}}, -3/2\right)$ are max pts for f on D

iv) D is connected, $f(D)$ is an interval

$$f(D) = [\min_{D} f , \max_{D} f]$$

$$= [-9/4 , 25/4] .$$

Do 3.6.7

Ex 3.6.9.

$$f(x,y) = (\underset{\uparrow}{x^2} + y^2)^2 - \underset{\uparrow}{3x^2y}$$

$$D = \mathbb{R}^2$$

i) $\lim_{(x,y)\to\infty_2} f$?

ii) det st. pts of f

iii) det min/max of f on $D = \mathbb{R}^2$ (if any)

iv) $f(\mathbb{R}^2)$?

Sol: i) $f(x,y) = (x^2+y^2)^2 - 3x^2y$

$$x = \rho\cos\theta$$
$$\underline{\underline{y}} = \rho\sin\theta \qquad (\rho^2)^2 - 3(\rho\cos\theta)^2\rho\sin\theta$$

$$= \rho^4 - 3\rho^3 \underbrace{\cos^2\theta\sin\theta}_{}$$

$$(x,y) \to \infty_2 \iff \rho \to +\infty, \quad 0 \leq \theta \leq 2\pi$$

$$\begin{tikzcd}
\rho^4 - 3\rho^3 \arrow[d] & \leq & f \arrow[d] & \leq & \rho^4 + 3\rho^3 \arrow[d, "\rho \to +\infty"] \\
+\infty & & +\infty & & +\infty
\end{tikzcd}$$

$$\Rightarrow \quad \exists \lim_{(x,y)\to\infty_2} f = +\infty.$$

ii) Let's check diff:

$$\left.\begin{aligned} \partial_x f &= 2(x^2+y^2)2x - 6xy \\ \partial_y f &= 2(x^2+y^2)2y - 3x^2 \end{aligned}\right\} \in \mathcal{C}(\mathbb{R}^2)$$

$\Rightarrow$ f is diff on $\mathbb{R}^2$.

(x,y) is st pt for f $\iff$

$$\nabla f(x,y) = \vec{0} \iff \begin{cases} \cancel{2}x\left[2(x^2+y^2) - 3y\right] = 0 \\ 4y\underline{(x^2+y^2)} - 3x^2 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ 4y^3 = 0 \end{cases} \quad \vee \quad \begin{cases} 2(x^2+y^2) - 3y = 0 \\ \Updownarrow \\ x^2+y^2 = \boxed{\frac{3}{2}y} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$(0,0)$$

$$\overset{2}{\cancel{4}}y \cdot \frac{\cancel{3}}{\cancel{2}}y - \cancel{3}x^2 = 0$$

$$\Updownarrow$$

$$\begin{cases} 2y^2 - x^2 = 0 \\ x^2+y^2 = \frac{3}{2}y \leftarrow \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x^2 = 2y^2 \\ \underbrace{2y^2+y^2}_{\cancel{3}y^2} = \frac{\cancel{3}}{\cancel{2}}y \end{cases}$$

$$\Leftrightarrow$$

$$\begin{cases} x^2 = 2y^2 \end{cases}$$

$$2y^2 - y = 0$$

$$\Updownarrow$$

$$y(2y-1) = 0$$

$$\Leftrightarrow \begin{cases} y = 0 \\ x^2 = 0 \end{cases} \quad \vee \quad \begin{cases} y = \tfrac{1}{2} \\ x^2 = 2 \cdot \frac{1}{4} = \tfrac{1}{2} \end{cases}$$

$$(0,0) \qquad \left(\pm \frac{1}{\sqrt{2}}, \tfrac{1}{2}\right)$$

Conclusion: st. pts are $(0,0)$, $\left(\pm \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

iii) $\underline{\text{Existence}}$ D is not compact (D is closed but it is unbded)

By i) $\nexists \max\limits_{\mathbb{R}^2} f$.

Since D is closed & unbded and

$$\lim_{\infty_2} f = +\infty$$

$$+ \quad f \in \mathcal{C}(D) \quad \Rightarrow \quad \exists \min_{D} f.$$

Search: if (x,y) is min pt for f on $D = \mathbb{R}^2 \Rightarrow (x,y) \in \underset{\underset{\mathbb{R}^2}{\|}}{\mathrm{Int}\,\mathbb{R}^2}$

$\Rightarrow$ (Fermat) $\nabla f(x,y) = \vec{0}$

$\Rightarrow$ (x,y) is one of the following pts

$$(0,0) \xrightarrow{f} 0$$

$$\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \xrightarrow{f} \left(\frac{1}{2}+\frac{1}{4}\right)^2 - 3\,\frac{1}{2}\cdot\frac{1}{2}$$

$$= \frac{9}{16} - \frac{3}{4}$$

$$= \boxed{-\frac{3}{16}}$$

$\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ are ^{gl} min pts for f on $\mathbb{R}^2$

iv) $\mathbb{R}^2$ is connected $\Rightarrow f(\mathbb{R}^2)$ interval

$$\begin{array}{l} \| \\ \left[-\frac{3}{16}, +\infty\right[. \end{array}$$

$\blacksquare$

Do Ex 3.6.12

3.6.10

3.6.11 / 3.6.8