Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \to \mathbb{R}.$

Pb: Determine $\min_D f$, $\max_D f$.

Notice that, for such f, increasing/decreasing do not make any sense

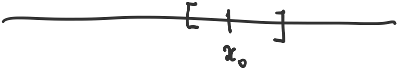
$$\left(\vec{x} \overset{?}{\leq} \vec{y} \implies f(\vec{x}) \leq f(\vec{y}) \right)$$

so we cannot expect that $\operatorname{sgn} f'$ will be of some interst

$$f'(\vec{x}) = \nabla f(\vec{x}) \overset{?}{\gtrless} \vec{0}$$

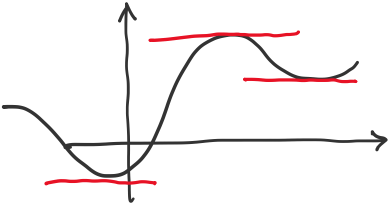
Let's remind that the first result on min/max for $f = f(x)$ $(x \in \mathbb{R})$ is the Fermat thm:

Thm: If $x_0 \in \operatorname{Int} D$ and f is diff at x_0,



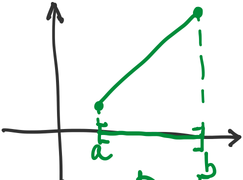
x_0 local min pt $\Longrightarrow$ $f'(x_0) = 0$

max



Rmk: if loc min/max $x_0 \in D \setminus \operatorname{Int} D$

$\Rightarrow$ $f'(x_0)$ is not necessarily $= 0$.



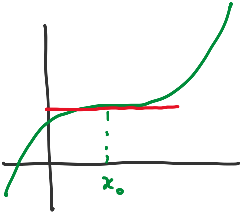
$D = [a, b]$

a is min pt

b is max pt

and $f'(a), f'(b) \neq 0$

Rmk: Even if $f'(x_0) = 0$, x_0 is not nec. a min nor a max



Fermat thm has a version for

$$f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

Thm: Let $\boxed{\vec{x}_0 \in \text{Int } D}$, f diff at $\vec{x}_0$

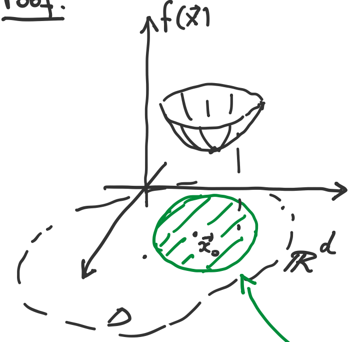
Then, if

$$\vec{x}_0 \text{ is local min/max for } f \text{ on } D$$
$$\Downarrow$$
$$\nabla f(\vec{x}_0) = \vec{0}.$$

Rmk: Of course, prev rmks remain valid:

- if $\vec{x}_0 \in D \setminus \text{Int } D \implies \nabla f$ could be $\neq 0$ ($\vec{x}_0$ loc min/max pt for f)
- $\nabla f(\vec{x}_0) = \vec{0} \not\Rightarrow \vec{x}_0$ is loc min/max.

Proof:



Assume, for inst, $\vec{x}_0$ be a loc. min:

$\exists B(\vec{x}_0, r]$:

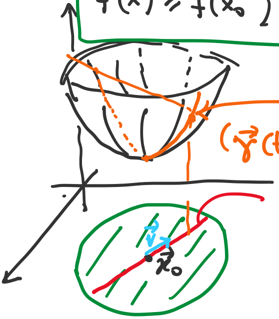
$$f(\vec{x}) \geq f(\vec{x}_0) \quad \forall x \in D \cap \overset{\overset{B(x_0,\rho]}{\cap}}{B(x_0,r]}$$

Since $\vec{x}_0 \in \text{Int } D$ $\quad \exists \underset{\underset{D}{\cap}}{B(\vec{x}_0, \hat{r}]}$

If $\quad \rho = \min\{r, \hat{r}\}$ $\quad \rho \leq \hat{r} \Rightarrow \underbrace{B(\vec{x}_0, \rho] \subset B(\vec{x}_0, \hat{r}] \subset D}$

$\rho \leq r \Rightarrow$

$$\boxed{f(\vec{x}) \geq f(\vec{x}_0) \quad \forall x \in D \cap B(\vec{x}_0, \rho]} = B(\vec{x}_0, \rho]$$



$f(\vec{\gamma}(t))$

$(\vec{\gamma}(t), f(\vec{\gamma}(t)))$

$\vec{x}_0 + t\vec{v} = \vec{\gamma}(t)$

Call $\quad \varphi(t) := f(\vec{\gamma}(t))$

$$\underset{\underset{\mathbb{R}}{\cap}}{t} \longmapsto \underbrace{f(\vec{\gamma}(t))}_{\in \mathbb{R}}$$

This φ has a min at $t=0$

(infact $\varphi(t) = f(\underbrace{\vec{\gamma}(t)}_{t \to 0})$ $\overset{\in}{} B(\vec{x}_0,\rho]$

$\vec{x}_0$ is loc min

$$\downarrow$$

$$\geq f(\vec{x}_0) = \underbrace{f(\vec{\gamma}(0))}_{} = \varphi(0)$$

$$\Rightarrow \quad \varphi(t) \geq \varphi(0) \qquad t \to 0$$

$\Rightarrow$ (Fermat for functs $\mathbb{R} \to \mathbb{R}$)

$$\boxed{\varphi'(0) = 0}$$

$$\varphi'(t) = \frac{d}{dt} f(\vec{\gamma}(t))$$

$$= \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

$$\vec{\gamma}(t) = \vec{x}_0 + t\vec{v}, \qquad \vec{\gamma}'(t) = \vec{v}$$

$$\left(\quad \vec{\gamma} = \begin{pmatrix} x_{0,1} + t v_1 \\ x_{0,2} + t v_2 \\ \vdots \\ x_{0,d} + t v_d \end{pmatrix} \qquad \vec{\gamma}'(t) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} = \vec{v} \right.$$

$$\Rightarrow \quad \varphi'(t) = \nabla f(\vec{\gamma}(t)) \cdot \vec{v}$$

$\varphi'(0) = 0$
$\Rightarrow$

$$\nabla f(\vec{x}_0) \cdot \vec{v} = 0 \quad \forall \vec{v} \in \mathbb{R}^d$$

$$\nabla f(\vec{x}_0) \perp \vec{v} \quad \forall \vec{v} \in \mathbb{R}^d$$

$$(\text{if } \vec{v} = \nabla f(\vec{x}_0) \Rightarrow \nabla f(\vec{x}_0) \cdot \nabla f(\vec{x}_0) = 0$$

$$\vec{v} \cdot \vec{v} = \sum v_j^2 = \|\vec{v}\|^2 \Rightarrow \|\nabla f(\vec{x}_0)\|^2 = 0$$

$$\Rightarrow \nabla f(\vec{x}_0) = \vec{0}. \quad \blacksquare$$

Def: A point $\vec{x}$ s.t. $\nabla f(\vec{x}) = \vec{0}$ is called stationary / critical point for f.

Do 3.6.4

Det the st. pts of

i) $f(x,y) := xy(x+1)$

Let's first check if f is diff.

$f = x^2 y + xy$

We apply the diff test :

$$\left.\begin{array}{l}\partial_x f = 2xy + y \\ \partial_y f = x^2 + x\end{array}\right. \in \mathcal{C}(\mathbb{R}^2) \qquad D = \mathbb{R}^2$$

$\Rightarrow$ f is diff on D.

$\underline{(x,y)}$ is st. pt for f $\Leftrightarrow$

$$\nabla f(x,y) = \vec{0}$$

$$\Leftrightarrow \quad (\partial_x f \,,\, \partial_y f) = (0,0)$$

$$\Leftrightarrow \quad \begin{cases} \partial_x f = 0 \\ \partial_y f = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} 2xy + y = 0 \\ \boxed{x^2 + x = 0} \end{cases} \qquad x(x+1) = 0$$

$$\Leftrightarrow \quad \boxed{\begin{cases} x = 0 \\ y = 0 \end{cases}} \quad \vee \quad \begin{cases} x = -1 \\ -2y + y = 0 \end{cases}$$

(0,0)

$$\Updownarrow$$

$$\begin{cases} x = -1 \\ y = 0 \end{cases}$$

(-1,0)

Conclusion: st pts are $(0,0)$, $(-1,0)$. ■

v) $f(x,y,z) = (x^3 - 3x - y^2)\, z^2 + z^3$

Sol: f is defd on $D = \mathbb{R}^3$.

To check diff we notice that

$$\left.\begin{aligned} \partial_x f &= (3x^2 - 3)\, z^2 \\ \partial_y f &= -2y z^2 \\ \partial_z f &= 2z(x^3 - 3x - y^2) + 3z^2 \end{aligned}\right\} \in \mathcal{C}(\mathbb{R}^3)$$

f is diff on $D = \mathbb{R}^3$

(x,y,z) is st pt for f $\Leftrightarrow$

$$\nabla f = \vec{0} \iff \begin{cases} (3x^2 - 3)\, z^2 = 0 \\ \boxed{-\cancel{2}yz^2 = 0} \\ 2z(x^3 - 3x - y^2) + 3z^2 = 0 \end{cases}$$

$$\iff \begin{cases} y = 0 \\ 3\boxed{(x^2 - 1)\, z^2 = 0} \\ 2z(x^3 - 3x) + 3z^2 = 0 \end{cases} \vee \begin{cases} z^2 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$$\Updownarrow$$

$$\Updownarrow \qquad \begin{cases} z = 0 \\ \\ \end{cases}$$

$$(x,y,0) \quad \forall\, x,y \in \mathbb{R}$$

$$\begin{cases} y = 0 \\ (x-1)(x+1)\,z^2 = 0 \\ z\left[2\left(x^3-3x\right)+3z\right] = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} y = 0 \\ x = 1 \\ z[-4+3z] = 0 \end{cases} \vee \begin{cases} y = 0 \\ x = -1 \\ z[4+3z] = 0 \end{cases} \vee \begin{cases} y = 0 \\ z^2 = 0 \\ 0 = 0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} y = 0 \\ x = 1 \\ z = 0 \end{cases} \vee \begin{cases} y = 0 \\ x = 1 \\ z = 4/3 \end{cases} \vee \begin{cases} y = 0 \\ x = -1 \\ z = 0 \end{cases} \vee \begin{cases} y = 0 \\ x = -1 \\ z = -4/3 \end{cases} \vee \begin{cases} y = 0 \\ z = 0 \end{cases}$$

$\underline{(1,0,0)}$ $(1,0,4/3)$ $(-1,0,0)$ $(-1,0,-4/3)$ $(x,0,0)$ $x \in \mathbb{R}$

Conclusion: st pts are:

- $(x,y,0)$ $\forall\, x,y \in \mathbb{R}$

- $(1,0,4/3)$, $(-1,0,-4/3)$.

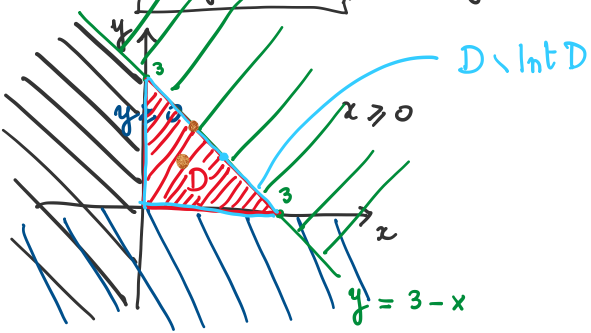
■

Example 3.3.5.

Det $\underline{\text{min/max}}$ for $f(x,y) = x + y - xy$

on $D = \{(x,y) \in \mathbb{R}^2 : \; 0 \leq y \leq 3-x, \, x \geq 0\}$

$\underline{\text{Sol}}$: $D = \{\underbrace{0 \leq y \leq 3 - x,}\; x \geq 0\}$



Since $f \in \mathcal{C}(D)$, D is closed (being defd by large inequalities) and bdded

(indeed:

$$\begin{cases} 0 \leq \boxed{y \leq 3-x} \\ x \geq 0 \end{cases} \quad \begin{array}{l} \Rightarrow \quad 0 \leq y \leq 3 \\ \Rightarrow \quad x \geq 0 \end{array}$$

$$\Big\downarrow$$

$$\begin{array}{l} x \leq 3-y \\ y \geq 0 \end{array} \quad \Rightarrow \quad x \leq 3$$

$$\Rightarrow \quad 0 \leq x, y \leq 3 \;\Big).$$

$\Rightarrow$ (W thm) f has both min/max on D.

If $(x,y) \in D$ is min/max pt for f on D

$\Rightarrow$ either $(x,y) \in \operatorname{Int} D \overset{F.}{\Rightarrow} \nabla f = \vec{0}$

or $(x,y) \in D \setminus \operatorname{Int} D$

If $(x,y) \in \operatorname{Int} D$, and since f is diff

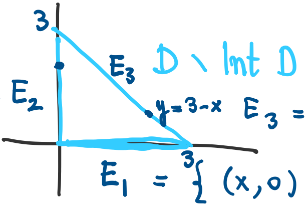
$$\left(\begin{matrix} \partial_x f = 1 - y \\ \partial_y f = 1 - x \end{matrix} \in \mathcal{C}(D) \Rightarrow \begin{matrix} f \text{ is diff} \\ \text{by diff test} \end{matrix} \right)$$

$$\Rightarrow (F) \qquad \nabla f = \vec{0} \iff \begin{cases} 1 - y = 0 \\ 1 - x = 0 \end{cases}$$

$$\iff \begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$\Rightarrow \quad (x,y) = (1,1).$$

If $(x,y) \in D \setminus \operatorname{Int} D = E_1 \cup E_2 \cup E_3$



$\{(0,y) : 0 \leq y \leq 3\} = E_2$

$E_3 = \{ (x, 3-x) : 0 \leq x \leq 3 \}$

$E_1 = \{ (x, 0) : 0 \leq x \leq 3 \}$

$$(x,y) \in E_1 \Rightarrow f(x,y) = f(x,0) = x$$

for min $\quad x = 0 \quad (0,0)$

for max $\quad x = 3 \quad (3,0)$

$$(x,y) \in E_2 \Rightarrow f(x,y) = f(0,y) = y$$

for min $y = 0 \Rightarrow (0,0)$

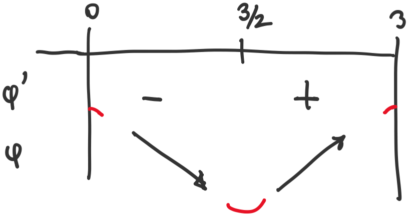
for max $y = 3 \Rightarrow (0,3)$

$$(x,y) \in E_3 \Rightarrow f(x,y) = f(x, 3-x)$$

$$= \cancel{x} + (3 - \cancel{x}) - x(3-x)$$

$$= \underbrace{3 - 3x + x^2}_{\varphi(x)} \qquad 0 \leq x \leq 3$$

$$\varphi'(x) = -3 + 2x \geq 0 \Leftrightarrow x \geq 3/2$$



for min $x = 3/2$

$$\Downarrow$$

$$\left(\frac{3}{2}, 3 - \frac{3}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}\right)$$

for max $x = 0, 3$

$$\Downarrow$$

$(0,3)$ $(3,0)$

Candidates for min

$$(1,1) \xrightarrow{f} 1$$

min pt $\boxed{(0,0)} \longrightarrow \boxed{0}$

$$\left(\tfrac{3}{2},\tfrac{3}{2}\right) \longrightarrow 3 - 9/4 = \frac{3}{4}$$

Cands for max

$$(1,1) \longrightarrow 1$$

$$\boxed{\begin{array}{l} (3,0) \\ (0,3) \end{array}} \begin{array}{l} \longrightarrow \boxed{3} \\ \longrightarrow \boxed{3} \end{array}$$

max pts.

Do Ex 3.6.6/7