Wednesday, 22 November 2023 08:36

$\vec{f}$ diff at $\vec{x}$ $\Leftrightarrow$ $\exists\ \vec{f}'(\vec{x})$ $m \times d$ matrix

$(\ \vec{f} = \vec{f}(\vec{x}) \in D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m\ )$

$$\boxed{\vec{f}(\overset{\vec{y}}{\vec{x} + \vec{h}}) = \underline{\vec{f}(\vec{x}) + \vec{f}'(\vec{x})\vec{h}} + \vec{o}(\vec{h})}$$

$$\Leftrightarrow \quad \boxed{\lim_{\vec{h} \to 0} \frac{\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) - \vec{f}'(\vec{x})\vec{h}}{\|\vec{h}\|} = \vec{0}.}$$

We proved that diffentiability is stronger than directional derivabilty

$$\exists\ \vec{f}'(\vec{x}) \quad \Rightarrow \quad \exists\ D_{\vec{v}}\vec{f}(\vec{x}) = \vec{f}'(x)\vec{v}$$

From this it follows

$$\vec{f}'(\vec{x}) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 - - & \partial_d f_1 \\ \partial_1 f_2 & \partial_2 f_2 - - .. & \partial_d f_2 \\ \vdots & & \\ \partial_1 f_m & \partial_2 f_m \cdots & \partial_d f_m \end{bmatrix}$$

$\vec{f} = (f_1, f_2, \ldots, f_m)$

If $m = 1$, $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$

$$f'(\vec{x}) = [\partial_1 f \cdots \cdots \quad \cdots \quad \partial_d f]$$

$$=: \nabla f$$

So, in gen, $\vec{f} = \vec{f}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$

$\vec{f} = (f_1 \cdots, f_m)$

$$\vec{f}'(\vec{x}) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} (\vec{x})$$

Prop: If

$\exists \vec{f}'(\vec{x}) \implies \vec{f}$ is cont at pt $\vec{x}$

Proof: We have to verify

$$\lim_{\vec{y} \to \vec{x}} \vec{f}(\vec{y}) = \vec{f}(\vec{x})$$

By differentiability of $\vec f$ at $\vec x$ we have

$$\vec f(\vec y) = \vec f(\vec x) + \vec f'(\vec x)\underbrace{(\vec y - \vec x)}_{\downarrow \, \vec 0} + \overset{\nearrow \vec 0}{\vec o(\vec y - \vec x)}$$

$\vec y = \vec x + \vec h$

$\vec h = \vec y - \vec x$

$\vec y \to \vec x$

$$\vec f'(\vec x)\,\vec 0 = \vec 0$$

$$\to \vec f(\vec x) \,.$$ ■

Rmk: Since $\exists f$ s.t.

- $\exists \; D_{\vec v} \vec f(\vec x) \quad \forall \vec v$

but $\vec f$ is not cont at $\vec x$

$\Rightarrow$ diff. is strictly stronger that dir. diff. ■

Rmk: Apparently, since to write $\vec f'(\vec x)$ we need $\partial_j f_i$ $\quad i = 1, \dots, m$, $j = 1, \dots, d$,

it seems that if $\exists \, \partial_j f_i \;\; \forall i, j$

then $\exists \vec{f}'(\vec{x})$.

However, this is false! In the case we mentioned in prev rmk

$$\exists D_{\vec{v}} f \quad \forall \vec{v} \quad \Rightarrow \quad \exists \partial_j f \quad \forall j$$

but that f cannot be diff. ■

Ex 3.2.3

Discuss diff at $\vec{0}$ of

$$f(x,y) := \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq \vec{0} \\ \vec{0} & (x,y) = \vec{0} \end{cases}$$

Sol: $f : D = \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\vec{0} \in \operatorname{Int} D$

The jac matrix is a one-line matrix

$$\nabla f = \left(\partial_x f(0,0), \ \partial_y f(0,0)\right).$$

$\partial_x f(0,0)$?

$$\underset{(x,y)\neq\vec{0}}{f(x,y) =} \frac{x^2y^2}{x^2+y^2}$$

$$\partial_x f = \frac{2xy^2(x^2+y^2) - x^2y^2\cdot 2x}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2}$$

But what about $(0,0)$?

By def

$$\partial_x f(0,0) = D_{(1,0)} f(0,0)$$

$$= \lim_{t\to 0} \frac{f\big(\overset{(t,0)}{\overbrace{(0,0)+t(1,0)}}\big) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\overset{0}{\overbrace{\frac{t^2\cdot 0^2}{t^2+0^2}}} - 0}{t}$$

$$= \lim_{t\to 0} \overset{0}{\overbrace{\frac{0}{t}}} = 0$$

By symmetry $\partial_y f(0,0) = 0$

$$\Rightarrow \nabla f(0,0) = (0,0) \equiv \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Now, by def f is diff at $\vec{0}$

if $$\boxed{\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{0}+\vec{h})-f(\vec{0})-f'(0)\vec{h}}{\|\vec{h}\|}} = 0$$

$$\Vert \qquad \vec{h}=(u,v)$$

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)-f(0,0)-\begin{bmatrix}0 & 0\end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix}}{\underbrace{\sqrt{u^2+v^2}}_{\|\vec{h}\|}}$$

$$\Vert$$

$$\lim_{(u,v)\to(0,0)} \frac{\overset{f(u,v)}{\frac{u^2v^2}{u^2+v^2}} - \overset{f(0,0)}{0} - 0}{\sqrt{u^2+v^2}}$$

$$\Vert$$

$$\lim_{(u,v)\to(0,0)} \underbrace{\boxed{\frac{u^2v^2}{(u^2+v^2)^{3/2}}}}_{g(u,v)} \overset{?}{=} 0 \qquad \text{YES!}$$

$$g \overset{\substack{u = \rho\cos\theta \\ v = \rho\sin\theta}}{=} \frac{\rho^{\cancel{4}}\cos^2\theta\,\sin^2\theta}{(\rho^2)^{3/2}}$$

$$= \cancel{\rho^3}$$

$$g = \rho\,\boxed{\cos^2\theta\,\sin^2\theta}$$

$$(u,v) \to (0,0) \iff \rho \to 0 \qquad 0 \leq \theta \leq 2\pi$$

$$\begin{tikzcd} 0 \arrow[d] & \leq & g & \leq & \rho \arrow[d] \\ 0 & & & & 0 \end{tikzcd}$$

$$\Rightarrow g \to 0 \qquad (u,v) \to (0,0)$$

$\Rightarrow$ f is diff at $(0,0)$ ■

Do Ex 3.6.2

#2. Diff at $\vec{0}$ of

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{|x|+|y|} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$$

Sol: f is defd on $D = \mathbb{R}^2$, $\vec{0} \in \text{Int}\, D$

First step: compute the jac matrix

$$f'(\vec{0}) = [\partial_x f(0,0) \quad \partial_y f(0,0)]$$

$$\partial_x f(0,0) \overset{\text{by def}}{=} D_{(1,0)} f(0,0)$$

$$= \lim_{t \to 0} \frac{f((0,0) + t(1,0)) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\overset{0}{\overset{\|}{f(t,0)}} - \overset{0}{\overset{\|}{f(0,0)}}}{t} = 0$$

Similarly $\partial_y f(0,0) = 0$

$\Rightarrow \quad \nabla f = (0, 0)$.

Second step: we have to verify if

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{0} + \vec{h}) - f(\vec{0}) - f'(\vec{0})\vec{h}}{\|\vec{h}\|} = 0$$

$$\shortparallel \quad \vec{h} = (u,v)$$

$$\lim_{(u,v)\to(0,0)} \frac{\frac{u^2v^2}{|u|+|v|} - 0 - \overset{\overset{0}{\shortparallel}}{\begin{bmatrix} 0 & 0 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}}}{\sqrt{u^2+v^2}}$$

$$\shortparallel$$

$$\lim_{(u,v)\to(0,0)} \underbrace{\frac{u^2v^2}{\sqrt{u^2+v^2}\,(|u|+|v|)}}_{g(u,v)} \overset{?}{=} 0$$

$$u = \rho\cos\theta$$
$$v = \rho\sin\theta$$

$$g = \frac{\rho^4 \cos^2\theta \sin^2\theta}{\rho\left(|\rho\cos\theta| + |\rho\sin\theta|\right)}$$

$$= \frac{\rho^{\overset{2}{\cancel{4}}} \cos^2\theta \sin^2\theta}{\cancel{\rho^2}\left(|\cos\theta| + |\sin\theta|\right)}$$

$$0 \leq g = \rho^2 \frac{\cos^2\theta \sin^2\theta}{|\cos\theta| + |\sin\theta|} \leq \rho^2 \frac{1}{|\cos\theta| + |\sin\theta|}$$

$$\frac{1}{|\cos\theta| + |\sin\theta|} \overset{?}{\leq} C \qquad \theta \in [0, 2\pi]$$

$$\Updownarrow$$

$$\underbrace{|\cos\theta| + |\sin\theta|}_{\overset{\|}{\varphi(\theta)}} \geq \frac{1}{C} > 0$$

$$\varphi \in \mathcal{C}([0,2\pi]) \Rightarrow (\text{W. thm}) \quad \exists\ \theta^*_{min}$$

$$\varphi(\theta) \geq \varphi(\theta^*_{min}) > 0$$

$$\Rightarrow \quad \begin{tikzcd} 0 \arrow[d] & \leq g \leq & C\rho^2 \arrow[d] \\ 0 & & 0 \end{tikzcd} \quad \Rightarrow \quad g \to 0.$$

∎

Rmk: If $f = f(\vec{x})$, f diff means

$$f(\vec{x}+\vec{h}) = f(\vec{x}) + \underbrace{f'(\vec{x})}\vec{h} + o(\vec{h})$$

$$\begin{bmatrix} \partial_1 f & \partial_2 f \dots & \partial_d f \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix}$$

$$\|$$

$$\partial_1 f\, h_1 + \partial_2 f\, h_2 + \dots + \partial_d f\, h_d$$

$$= (\partial_1 f, \partial_2 f, \dots, \partial_d f) \underset{\uparrow}{\cdot} (h_1 \text{---} h_d)$$

scalar prod

$$= \nabla f \cdot \vec{h}$$

that is, for $\mathbb{R}$-valued functs, f is diff at $\vec{x}$ $\Longleftrightarrow$

$$f(\vec{x}+\vec{h}) = f(\vec{x}) + \nabla f(\vec{x})\cdot\vec{h} + o(\vec{h})$$

$\blacksquare$

Thm (differentiability test)

$\vec{f} = \vec{f}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$,

$$\vec{f} = \underbrace{(f_1, \ldots, f_m)}_{\text{components of } f.}$$

If

$$\partial_j f_i \in \mathcal{C}(D) \qquad \forall\, \begin{array}{l} j = 1 \underline{\quad} d \\ i = 1 \underline{\quad} m \end{array}$$

$\Rightarrow$ f is diff at every $\vec{x} \in D$.

Ex: $f(x,y) = xe^y + \sin(x^2+y^2)$.

This f is defd on $D = \mathbb{R}^2$

$$\partial_x f = e^y + \cos(x^2+y^2)\cdot 2x$$

$$\partial_y f = xe^y + \cos(x^2+y^2)\cdot 2y \qquad \forall (x,y) \in \mathbb{R}^2$$

and since $\partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow$
f is differentiable on $\mathbb{R}^2$. ∎

Rmk: What if we apply diff test to one of prev exercises, for inst to Ex 3.2.3

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0}. \end{cases}$$

Sol: To apply diff test we should verify whether $\partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2)$ or not.

$$\partial_x f(x,y) = \begin{cases} \dfrac{2xy^4}{(x^2+y^2)^2} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$$

$\partial_x f \in \mathcal{C}(\mathbb{R}^2 \setminus \{(0,0)\})$ (ratio of cont functs, den $\neq 0$)

What about $(0,0)$?

$\partial_x f$ cont at $(0,0)$ $\Leftarrow$

$$\lim_{(x,y)\to(0,0)} \partial_x f(x,y) \overset{?}{=} \partial_x f(0,0) = 0$$

$$\|$$

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{2xy^4}{(x^2+y^2)^2}}_{\partial_x f(x,y)}$$

$$\partial_x f(x,y) \overset{\substack{x = \rho\cos\theta \\ y = \rho\sin\theta}}{=} \frac{2\rho\cos\theta(\rho\sin\theta)^4}{(\rho^2)^2}$$

$$= 2\,\frac{\rho^{\cancel{5}}\cos\theta\sin^4\theta}{\cancel{\rho^4}}$$

$$\partial_x f = 2\rho\boxed{\cos\theta\sin^4\theta}$$

$$\begin{array}{ccccc} -2\rho & \leq & \partial_x f & \leq & 2\rho\cdot 1 \\ \downarrow & & & & \downarrow \\ 0 & & & & 0 \end{array}$$

$$\begin{array}{c} \rho\to 0 \\ \Updownarrow \\ (x,y)\to(0,0) \end{array}$$

$$\Rightarrow \quad \partial_x f \longrightarrow 0 = \partial_x f(0,0)$$

$$\Rightarrow \partial_x f \in \mathcal{C}(\mathbb{R}^2)$$

Similarly $\partial_y f \in \mathcal{C}(\mathbb{R}^2)$ $\quad \Longrightarrow f$ diff on $\mathbb{R}^2$

(so in part at $(0,0)$) ∎

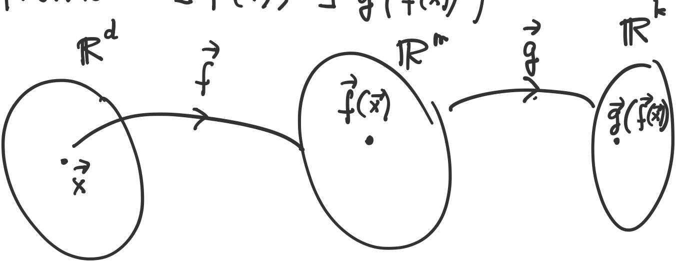
Do ex 3.6.3.

Chain Rule

Recall that

$$[g(f(x))]' = g'(f(x))\, f'(x)$$

(provided $\exists f'(x)$, $\exists g'(f(x))$)



Thm: If $\exists \vec{f}'(\vec{x})$, $\exists \vec{g}'(\vec{f}(\vec{x}))$

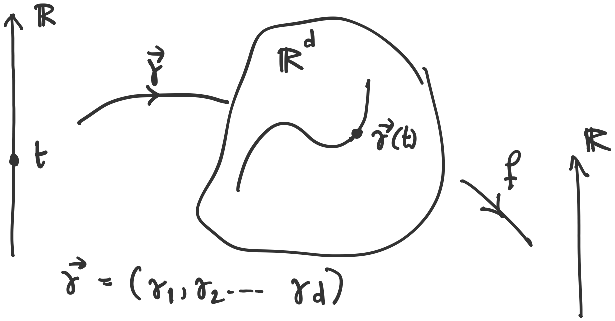
$$\Rightarrow \quad \exists \left[\vec{g}(\vec{f}(\vec{x}))\right]' = \vec{g}'(\vec{f}(\vec{x}))\, \vec{f}'(\vec{x})$$

$$\underbrace{\qquad\qquad}_{k \times \boxed{m}} \underbrace{\qquad\quad}_{\boxed{m} \times d}$$

A particular case of this is the so called total derivative :

$$\vec{\gamma}(t) = \vec{\gamma} : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d$$

$$f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$



$$\vec{\gamma} = (\gamma_1, \gamma_2 \cdots \gamma_d)$$

$$\frac{d}{dt} f(\vec{\gamma}(t)) = f'(\vec{\gamma}(t))\, \vec{\gamma}'(t)$$

$$= \left[\partial_1 f(\vec{\gamma}(t)) \quad \partial_2 f \ \cdots \ \partial_d f\right] \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_d'(t) \end{bmatrix}$$

$$= \partial_1 f(\vec{\gamma}(t))\, \gamma_1'(t) + \partial_2 f\, \gamma_2' + \partial_3 f\, \gamma_3' + \cdots$$

$$+ \partial_d f\, \gamma_d'$$

$$= (\partial_1 f \;\; \partial_2 f \cdots \;\; \partial_d f) \cdot (\gamma_1', \gamma_2', \cdots \;\; \gamma_d')$$

$$= \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

$$\boxed{\frac{d}{dt} f(\vec{\gamma}(t)) = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)}$$