Monday, 20 November 2023 08:37

Ex 3.6.1

iv) $D_{(-1,1)} f(0,0)$ $\qquad f(x,y) := \begin{cases} \dfrac{xy}{x^2+y^4} & (x,y)\neq\vec{0} \\ 0 & (x,y)=\vec{0} \end{cases}$

Sol: According to the def

$$D_{(-1,1)} f(0,0) = \lim_{t\to 0} \frac{f(\overset{(-t,t)}{\overset{\shortparallel}{(0,0)+t(-1,1)}}) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(-t,t) - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{t}$$

$$= \lim_{t\to 0} \frac{\dfrac{-t\cdot\cancel{t}}{(-t)^2+t^4}}{\cancel{t}}$$

$$= \lim_{t\to 0} -\frac{\cancel{t}\cdot 1}{t^{\cancel{2}} + t^{\cancel{4}\,3}} =$$

$$= \lim_{t\to 0} -\frac{1}{t+t^3} \quad \nexists \qquad \blacksquare$$

Partial Derivatives

$$D_{\vec{e}_j} f(\vec{x}) =: \partial_j f(\vec{x}) = \frac{\partial f}{\partial x_j}$$

$$\vec{e}_j = (0,0,\ldots 0,\underset{j}{1},0\ldots,0)$$

$$\left(\frac{df}{dx}\right)$$

$\uparrow$

partial deriv
with respect to
the j-th var.

$$\partial_j f(\vec{x}) \underset{\uparrow}{=} \lim_{t\to 0} \frac{f(\vec{x}+t\vec{e}_j) - f(\vec{x})}{t}$$

by def

If, for inst, f is
$f = f(x,y)$

$$\partial_x f(x,y) = D_{(1,0)} f(x,y)$$

$$= \lim_{t\to 0} \frac{f\big(\overset{(x,y)+(t,0)}{(x,y)+t(1,0)}\big) - f(x,y)}{t}$$

$$= \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t}$$

$$= \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t}$$

ordinary (1 var) derivative of f resp to x with y a const.

$$f(x,y) = x \cos y$$

$$\partial_x f = \partial_x (x \cos y)$$

$$= \partial_x (k x) = k = \cos y$$

$$\partial_y f = \partial_y (x \cos y) = x \, \partial_y \cos y$$

$$= - x \sin y .$$

Ex: $f(x,y,z) = xy + z^2 x + y e^z$

$$\partial_x f(x,y,z) = \partial_x (xy) + \partial_x (z^2 x) + \partial_x (y e^z)$$

$$= y \cdot 1 + z^2 \cdot 1 + 0$$

$$= y + z^2$$

$$\partial_y f = x + 0 + e^z$$

$$\partial_z f = 0 + 2zx + y e^z$$

■

Ex:

$$f(x,y) = \begin{cases} \boxed{\dfrac{xy}{x^2+y^2}} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$$

$\partial_x f(0,0)$, $\partial_y f(0,0)$?

(do they exist? if yes, what are their values?)

Sol: Naive sol:

$$\partial_x f = \boxed{\partial_x} \frac{xy}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$= \frac{y\cdot(x^2+y^2) - \overset{2x^2y}{xy\cdot 2x}}{(x^2+y^2)^2} = \frac{y^3 - x^2y}{(x^2+y^2)^2}$$

$$\left(\frac{N}{D}\right)' = \frac{N'D - N\cdot D'}{D^2}$$

?

Can I plug $\begin{matrix} x=0 \\ y=0 \end{matrix}$ in

No!

Can I say that since $\partial_x f(0,0)$ is not def'd $\Rightarrow \nexists\, \partial_x f(0,0)$?

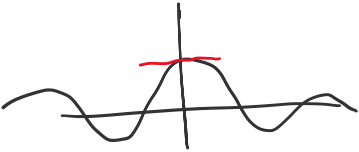
No!

$$\partial_x f(0,0) \overset{?}{=} \partial_x 0 = 0 \quad ?$$

No!

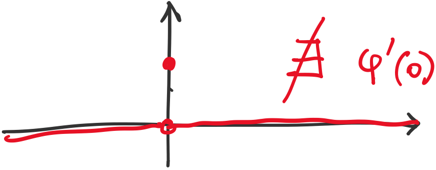
⌜

$$\varphi(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$



$$\varphi'(x) \underset{x=0}{=} \Big(\underset{\substack{\| \\ 1}}{\varphi(0)}\Big)' = 0 \quad ?$$

$$\varphi(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 1 \end{cases}$$



$\nexists \ \varphi'(0)$

$$\varphi'(x) = \Big(\underset{\substack{\| \\ 1}}{\varphi(0)}\Big)' = 0 \quad ??$$

⌟

So, how can we discuss $\partial_x f(0,0)$, $\partial_y f(0,0)$?

$$\partial_x f(0,0) = D_{(1,0)} f(0,0)$$

$$= \lim_{t\to 0} \frac{f\left(\overbrace{(0,0) + t(1,0)}\right) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{\overset{\boxed{\frac{t\cdot 0}{t^2+0^2}} = 0}{f(t,0)} - \overset{\overset{0}{\parallel}}{f(0,0)}}{t}$$

$$= \lim_{t\to 0} \underset{\underset{0}{\parallel}}{\frac{0}{t}} = 0$$

$$\Rightarrow \quad \exists \; \partial_x f(0,0) = 0$$

and, similarly (or by symmetry) $\partial_y f(0,0) = 0$.

■

Differential (the true def of deriv)

We introduced the dir. deriv., but this turns out to be a weak def of deriv for an $\vec{f} = \vec{f}(\vec{x})$.

So we still have the pb of def

$$\vec{f}\,'(\vec{x}) = \lim_{\vec{h}\to\vec{0}} \frac{\vec{f}(\vec{x}+\vec{h})-\vec{f}(\vec{x})}{\vec{h}}$$

(we cannot use this formula to def $\vec{f}\,'(\vec{x})$ because we cannot divide by $\vec{h}$)

Restarting for ord deriv for $f=\underset{\in\mathbb{R}}{f(x)}$ $x\in\mathbb{R}$

$$\cancel{f'(x)}\;0 = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} - f'(x)\right]$$

$$\Updownarrow$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)-f'(x)h}{h} = 0$$

Recalling that when

$$\lim_{h\to 0}\frac{\varphi(h)}{\psi(h)} = 0 \iff \varphi(h) = o\big(\psi(h)\big)$$

$$f(x+h)\overbrace{-f(x)-f'(x)h} = o(h)$$

$$\Updownarrow$$

$$\boxed{f(\overset{x}{\overbrace{x+h}}) = \overset{x_0}{f(x)} + \underbrace{f'(x)h} + \underbrace{o(h)}} \leftarrow$$

$$\left(f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0) \right.$$

$$\left. \qquad\qquad\qquad \text{Taylor formula} \right)$$

" Adding arrows "

$$\underbrace{\vec f(\vec x+\vec h)} \textcircled{=} \underbrace{\vec f(\vec x) + \underbrace{\vec f\,'(\vec x)}\,\vec h + \vec o(\vec h)}_{?}$$

Can we give a meaning to this formula?

$$\vec f = \vec f(\vec x) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$$

$$\underset{\in\mathbb{R}^m}{\vec f(\vec x+\vec h)} = \underset{\in\mathbb{R}^m}{\vec f(\vec x)} + \underbrace{\bigcirc\!\!\!\!\!\vec f\,'(\vec x)\,\overset{\overset{\mathbb{R}^d}{\downarrow}}{\vec h}}_{\in\mathbb{R}^m} + \vec o(\vec h)$$

$$m\left\{\overbrace{\begin{bmatrix}\cdot&\cdot&\cdot&\cdot&\cdot\\ &&&&\\ &&&&\\ &&&&\end{bmatrix}}^{d}\right. \overset{\vec h}{\begin{pmatrix}h_1\\h_2\\\vdots\\h_d\end{pmatrix}} = \left.\begin{pmatrix}-\\-\\-\\\vdots\\-\end{pmatrix}\right\}\begin{matrix}m\\\text{comp.}\end{matrix}$$

$m \times d$ matrix

What is $\vec{o}(\vec{h})$?

In principle,

$$\vec{\varphi}(\vec{h}) = o(\vec{h}) \Leftrightarrow \frac{\vec{\varphi}(\vec{h})}{\vec{h}} \xrightarrow[\vec{h}\to 0]{} \vec{0}$$

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{\varphi}(\vec{h})}{\vec{h}} = \vec{0}$$

but since we cannot divide by $\vec{h}$, we cannot use this as def.

Idea: $\vec{\varphi}(\vec{h}) = \vec{o}(\vec{h}) \Leftrightarrow \dfrac{\vec{\varphi}(\vec{h})}{\|\vec{h}\|} \to \vec{0}$

Def: We say that $\vec{\varphi}(\vec{h}) = \vec{o}(\vec{h})$ if

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{\varphi}(\vec{h})}{\|\vec{h}\|} = \vec{0}.$$

We are now ready to define the concept of differentiable function:

Def: Given $\vec{f} = \vec{f}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, we say that $\vec{f}$ is differentiable at pt. $\vec{x} \in \text{Int } D$ if there exists an $m \times d$ matrix (denoted by $\vec{f}'(\vec{x})$) such that

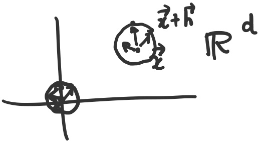
$$\vec{f}(\vec{x}+\vec{h}) = \underbrace{\vec{f}(\vec{x}) + \vec{f}'(\vec{x})\vec{h}}_{} + \vec{o}(\vec{h})$$

$$\Updownarrow$$

$$\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) - \underbrace{\vec{f}'(\vec{x})\vec{h}}_{} = \vec{o}(\vec{h})$$

$$\Updownarrow$$

$$\boxed{\lim_{\vec{h}\to\vec{0}} \frac{\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) - \vec{f}'(\vec{x})\vec{h}}{\|\vec{h}\|} = \vec{0}}$$



First pb: how do we compute $\vec{f}'(\vec{x})$?

$\vec{f}'(\vec{x})$ is called jacobian matrix

Prop: If $\vec{f}$ is diff at $\vec{x}$

$$\Rightarrow \quad \exists \; D_{\vec{v}} \vec{f}(\vec{x}) = \vec{f}'(\vec{x})\vec{v} \qquad \forall \vec{v} \in \mathbb{R}^d$$

In part.

$$\vec{f}'(\vec{x}) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 & \partial_3 f_1 \cdots & \partial_d f_1 \\ \partial_1 f_2 & \partial_2 f_2 & \cdots & \partial_d f_2 \\ \vdots & & & \\ \partial_1 f_m & \partial_2 f_m & \cdots & \partial_d f_m \end{bmatrix} (\vec{x})$$

$$\vec{f}(\vec{x}) = \left(f_1(\vec{x}), f_2(\vec{x}), \cdots, f_m(\vec{x}) \right)$$

Proof:

$$D_{\vec{v}} \vec{f}(\vec{x}) \underset{\substack{\text{by def} \\ \text{dir der}}}{=} \lim_{t \to 0} \frac{\boxed{\vec{f}(\vec{x} + \overbrace{t\vec{v}}^{\vec{h}}) - \vec{f}(\vec{x})}}{t}$$

By hypothesis:

$$\vec{f}(\vec{x} + \vec{h}) = \underbrace{\vec{f}(\vec{x})} + \vec{f}'(\vec{x})\vec{h} + \vec{o}(\vec{h})$$

$$\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) = \vec{f}'(\vec{x})\vec{h} + \vec{o}(\vec{h})$$

If $\vec{h} = t\,\vec{v}$

jac matrix

$$\vec{f}(\vec{x}+\underbrace{t\vec{v}}) - \vec{f}(\vec{x}) = \vec{f}'(\vec{x})[t\vec{v}] + \vec{o}(t\vec{v})$$

$$= \boxed{t\,\vec{f}'(\vec{x})\vec{v} + \vec{o}(t\vec{v})}$$

$$\Rightarrow D_{\vec{v}}\vec{f}(\vec{x}) = \lim_{t\to 0}\frac{t\,\vec{f}'(\vec{x})\vec{v} + \vec{o}(t\vec{v})}{t}$$

$$= \lim_{t\to 0}\left[\vec{f}'(\vec{x})\vec{v} + \frac{\vec{o}(t\vec{v})}{t}\right]$$

$$= \vec{f}'(\vec{x})\vec{v} + \lim_{t\to 0}\frac{\vec{o}(t\vec{v})}{t} \overset{?}{=} \vec{0}$$

$$= \vec{f}'(\vec{x})\vec{v}.$$

$$\frac{\vec{o}(t\vec{v})}{\|t\vec{v}\|} \longrightarrow 0 \qquad t\vec{v} \to \vec{0} \Longleftarrow t \to 0$$

$\|$

$$\frac{\vec{o}(t\vec{v})}{|t|\,\|\vec{v}\|} \to 0 \iff \frac{\vec{o}(t\vec{v})}{\cancel{|}t\cancel{|}} \to 0$$

$$\Rightarrow \quad \exists\ D_{\vec{v}}\vec{f}(\vec{x}) = \vec{f}'(\vec{x})\,\vec{v}$$

If now $\vec{f} = (f_1 - - - - f_m)$

$$M = \vec{f}'(\vec{x}) \qquad M = \begin{bmatrix} a_{11} & a_{12} \cdots & & a_{1d} \\ a_{21} & a_{22} & - - & a_{2d} \\ \vdots & & \boxed{a_{ij}} & \\ a_{m1} & a_{m2} & & a_{md} \end{bmatrix}$$

$$a_{ij} = \underbrace{M\vec{e}_j}_{j\text{-th col. of } M} \overset{\text{scalar prod}}{\cdot} \vec{e}_i$$

$$= \underbrace{\vec{f}'(\vec{x})\,\vec{e}_j} \cdot \vec{e}_i$$

$$= \left(D_{\vec{e}_j}\vec{f}(\vec{x})\right) \cdot \vec{e}_i$$

$$= D_{\vec{e}_j}\left(\underbrace{\vec{f}(\vec{x}) \cdot \vec{e}_i}_{f_i(\vec{x})}\right)$$

$$a_{ij} = D_{\vec{e}_j} f_i(\vec{x}) = \partial_j f_i$$

$$\begin{bmatrix} a_{11} & \text{———} & a_{1d} \\ a_{21} & \text{———} & a_{2d} \\ \vdots & & \\ a_{m1} & \text{———} & a_{md} \end{bmatrix} = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 & \text{———} & \partial_d f_1 \\ \partial_1 f_2 & \partial_2 f_2 & \text{———} & \partial_d f_2 \\ \vdots & & & \end{bmatrix}$$

Def: If $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \underline{\underline{\mathbb{R}}}$

$f'(\vec{x})$ is an $1 \times d$

$$\begin{bmatrix} \partial_1 f & \partial_2 f & \cdot \text{---} \cdot & \partial_d f \end{bmatrix} =: \nabla f(\vec{x})$$

gradient of f