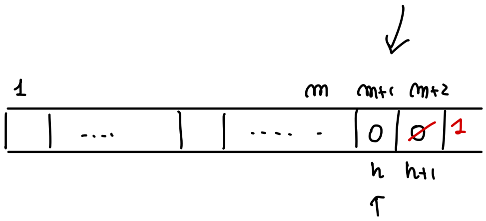
COMPUTABILITY (20/11/2023)

* EXERCISE : URM^P instructions

$Z(m)$
$T(m,n)$
$J(m,n,t)$
$\cancel{S(n)}$ $\qquad P(n)$ $\qquad r_n \leftarrow r_n \dot{-} 1 = \begin{cases} 0 & r_n = 0 \\ r_n - 1 & \text{if } r_n > 0 \end{cases}$

$\mathcal{C}^P \subsetneq \mathcal{C}$

($\mathcal{C}^P \subseteq \mathcal{C}$)
given a program P in URM^P



t: $\cancel{P(n)}$ $J(1,1,\text{SUB})$

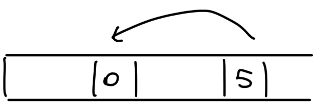
SUB: $J(n, m+1, t+1)$
$S(m+2)$
LOOP: $J(n, m+2, \text{RES})$
$S(m+1)$
$S(m+2)$
$J(1,1,\text{LOOP})$

RES: $T(m+1, n)$
$J(1,1,t+1)$

FORMAL PROOF : using $Z(n), S(n), T(m,n), J(m,n,t), P(n)$

For every program P ~~of URM^P~~ , for every $k \in \mathbb{N}$, there is a URM-program P' s.t. $f_{P'}^{(k)} = f_P^{(k)}$. (proof by induction on the number of predecessors)

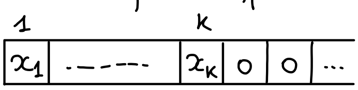
($\mathcal{C} \not\subseteq \mathcal{C}^P$)



Given a program P of URM^P and $\vec{x} \in \mathbb{N}^k$ the maximum value in memory after any number of steps of computation of $P(\vec{x})$ is $\leq \max\limits_{1 \leq i \leq n} x_i$

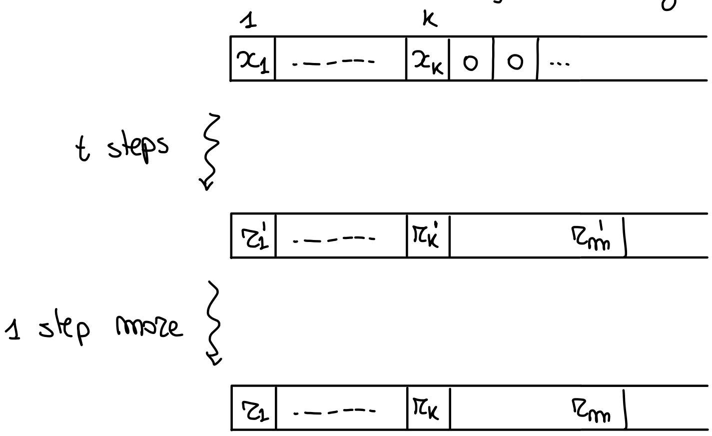
Proof by induction on the number t of computation steps.

($t=0$) the memory is



$\max\limits_i r_i = \max\limits_{1 \leq i \leq k} x_i$

$(t \to t+1)$ the content of memory after $t+1$ steps is



by inductive hyp

$$\max_i r_i' \leq \max_{1 \leq i \leq k} x_i$$

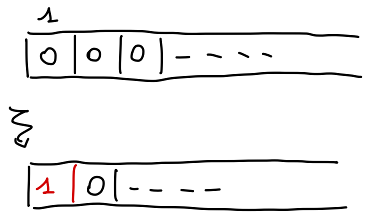
several cases according to the instruction executed at step $t+1$

$Z(n)$
$T(m, n)$
$J(m, n, t)$
$P(n)$

$$\max_i r_i \leq \max_i r_i' \leq \max_{1 \leq i \leq k} x_i$$

$\square$

* The successor $S: \mathbb{N} \to \mathbb{N}$ $\quad S(x) = x+1$ is not URM^P-computable



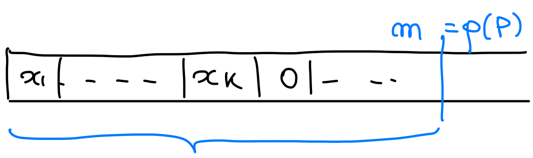
$\max r_i = 0$

$\wedge$

$\max r_i' = 1$ impossible

<u>NOTE</u>: Termination is decidable for the URM^P model

(EXERCISE)



(ASSIGNED)

<u>EXERCISE</u> : Show that there is a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that

$$E_{k(x)} = W_x$$

$P_x \rightsquigarrow P_{k(x)}$ with set of outputs = set of inputs where P_x terminates

$$\boxed{\begin{array}{l} \text{def } P_{k(x)}(y): \\ \quad P_x(y) \\ \quad \text{return } y \end{array}}$$

define $f : \mathbb{N}^2 \to \mathbb{N}$

$$f(x,y) = \begin{cases} y & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}(\varphi_x(y)) \cdot y \qquad\qquad \mathbb{1}(x) = 1 \quad \forall x$$

$\hookrightarrow$ 1 if $\varphi_x(y)\downarrow$, $\uparrow$ otherwise

$$= \mathbb{1}(\psi_U(x,y)) \cdot y$$

computable (composition of computable functions)

Hence by the smn - theorem there is $k : \mathbb{N} \to \mathbb{N}$ total and computable s.t.

$$\varphi_{k(x)}(y) = f(x,y) = \begin{cases} y & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

* k is the desired function $W_x = E_{k(x)}$

$(W_x \subseteq E_{k(x)})$ let $y \in W_x \Rightarrow \varphi_x(y)\downarrow \Rightarrow \varphi_{k(x)}(y) = f(x,y) = y$

hence $y \in E_{k(x)}$

$(E_{k(x)} \subseteq W_x)$ let $y \in E_{k(x)}$ i.e. there $z \in \mathbb{N}$ s.t. $\underbrace{\varphi_{k(x)}(z)}_{= f(x,z)} = y$

$\Rightarrow z = y$ and $\varphi_x(y)\downarrow \Rightarrow y \in W_x$

$\square$

<u>EXERCISE</u> : there is a total computable function $k : \mathbb{N} \to \mathbb{N}$ st.

$$W_{k(x)} = \mathbb{P} \qquad (\mathbb{P} \text{ set of even numbers})$$

$$E_{k(x)} = \{ y \in \mathbb{N} \mid y \geq x \}$$

define $f : \mathbb{N}^2 \to \mathbb{N}$

$$f(x,y) = \begin{cases} x + \underbrace{y/2}_{\mathbb{N}} & y \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

$$= x + qt(2,y) + \underbrace{\mu w.\ rm(2,y)}_{\substack{0 \quad \text{when } y \text{ is even} \\ \uparrow \quad \text{otherwise}}}$$

computable

Hence, by the smn theorem there is $k : \mathbb{N} \to \mathbb{N}$ total computable such that

$$\varphi_{k(x)}(y) = f(x,y) = \begin{cases} x + y/2 & \text{if } y \text{ even} \\ \uparrow & \text{otherwise} \end{cases}$$

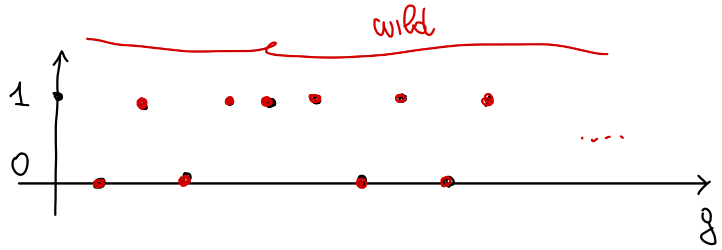
k is the desired function

→ $W_{k(x)} = \mathbb{P}$ ok

→ $E_{k(x)} \overset{?}{=} \{ y \mid y \geq x \}$

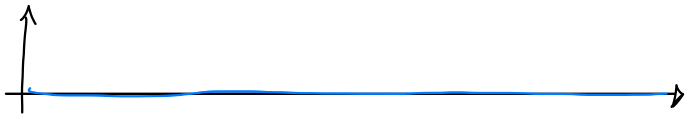
$$\begin{aligned} E_{k(x)} &= \{ \varphi_{k(x)}(y) \mid y \in W_x \} \\ &= \{ \varphi_{k(x)}(y) \mid y \in \mathbb{P} \} \\ &= \{ \varphi_{k(x)}(2z) \mid z \in \mathbb{N} \} \\ &= \{ x + \tfrac{2z}{2} \mid z \in \mathbb{N} \} \\ &= \{ x + z \mid z \in \mathbb{N} \} \\ &= \{ y \mid y \geq x \} \end{aligned}$$

EXERCISE : Are there f, g f computable, g not computable   s.t. $f \circ g$ computable ?



$$g(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

non computable



$$f(x) = 0 \quad \forall x$$

$$f(g(x)) = 0 \quad \forall x \qquad \text{computable}$$
$$= f(x)$$

* Are there f, g f not computable, g not computable   s.t. $f \circ g$ computable ?

* $g(x) = \begin{cases} 1 & \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$ not computable

* $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \varphi_x(x)+1 & \text{if } x > 1 \text{ and } \varphi_x(x)\downarrow \\ 0 & \text{if } x > 1 \text{ and } \varphi_x(x)\uparrow \end{cases}$

not computable

$$f \neq \varphi_x \qquad \forall x > 1$$

($\forall y \quad \exists x > 1$ s.t. $\varphi_x = \varphi_y \rightsquigarrow f \neq \varphi_x = \varphi_y$

$\rightsquigarrow$ f different from all computable functions)

but

$$f(g(x)) = 0 \quad \forall x \qquad \text{computable} \quad !$$

EXERCISE : Show that every computable function f can be obtained as the composition of two non computable functions g, h.

(ASSIGNED)

EXERCISE: Prove that $pow_2 : \mathbb{N} \to \mathbb{N}$, $pow_2(x) = 2^x$ is PR

by using only the definition of PR.

(least class of functions including basic functions (zero, successor, projections) and closed under composition, primitive recursion)

$x+y$ $\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases}$

$x*y$ $\begin{cases} x*0 = 0 \\ x*(y+1) = (x*y)+x \end{cases}$

x^y $\begin{cases} x^0 = 1 = succ(0) \\ x^{y+1} = (x^y)*x \end{cases}$

$succ : \mathbb{N} \to \mathbb{N}$
$succ(x) = x+1$

$pow2(x) = 2^y = succ(succ(0))^y$

alternatively

$$\begin{cases} pow_2(0) = 2^0 = 1 = succ(0) \\ pow_2(y+1) = 2^{y+1} = 2^y + 2^y = pow2(y) + pow2(y) \end{cases}$$

and observe that $+$ is in PR

alternatively

$$\begin{cases} pow_2(0) = 2^0 = 1 = succ(0) \\ pow_2(y+1) = 2^{y+1} = twice(2^y) \end{cases}$$

$twice : \mathbb{N}^1 \to \mathbb{N}^1$

$$\begin{cases} twice(0) = 0 \\ twice(y+1) = twice(y) + 2 = succ(succ(twice(y))) \end{cases}$$

EXERCISE: $\chi_{\mathbb{P}} \in PR$

$$\chi_{\mathbb{P}}(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{\mathbb{P}}(x) = \overline{sg}(rm(2, x))$$

$\vdots$ $\nwarrow$ complex!

directly

$$\begin{cases} \chi_{\mathbb{P}}(0) = 1 \\ \chi_{\mathbb{P}}(y+1) = \overline{sg}(\chi_{\mathbb{P}}(y)) \end{cases}$$

$$\begin{cases} \overline{sg}(0) = 1 \\ \overline{sg}(y+1) = 0 \end{cases}$$