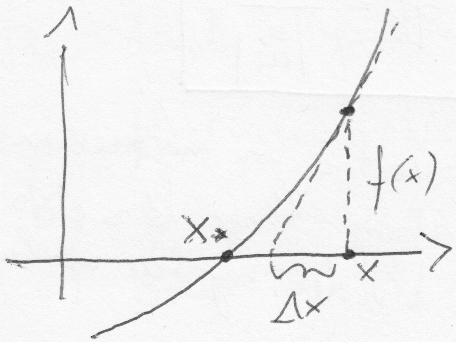
FINDING ROOTS. NEWTON'S METHOD: ERROR

i)



$$f'(x) = \frac{\Delta f(x)}{\Delta (x)} \Rightarrow \boxed{\Delta x = \frac{f(x)}{f'(x)}}$$

Iterate: $\boxed{x' = x - \Delta x = x - \frac{f(x)}{f'(x)}}$

ii) The error ε is, by definition $\boxed{x - x^* \equiv \varepsilon}$

•) Taylor expand around ~~x_*~~ x:

$$f(x_*) = f(x) \overset{\text{minus}}{\cancel{+}} f'(x) \underbrace{(x - x_*)}_{\varepsilon} + O(\varepsilon^2)$$

•) At the next step we get x' and $\boxed{\varepsilon' \equiv x' - x_*}$. Expand:

$$f(x_*) = f(x') \cancel{+} f'(x') \underbrace{(x' - x_*)}_{\varepsilon'} + O(\varepsilon'^2)$$

•) Remember that $\boxed{f(x_*) = 0}$. Take the first expansion:

$$\rightarrow f(x) + f'(x)(x - x_r) + O(\varepsilon^2) = 0$$

$$\frac{1}{2} f''(x)(x - x_r)^2$$

$$f(x) + f'(x)(x - x_r) + \frac{1}{2} f''(x)(x - x_r)^2 = 0$$

Now divide by $f'(x)$: $\quad \dfrac{f(x)}{f'(x)} + (x - x_r) + \dfrac{1}{2}\dfrac{f''(x)}{f'(x)}(x - x_r)^2 = 0$

$$\Rightarrow x_r = \underbrace{\left[x - \frac{f(x)}{f'(x)}\right]}_{\substack{\text{Newton's} \\ \text{rule} \\ \text{to get } x'}} + \underbrace{\frac{1}{2}\frac{f''(x)}{f'(x)}(x_r - x)^2}_{\substack{\text{Error, at leading} \\ \text{order}}} + \dots$$

Therefore: $\quad x_r = \underset{\substack{\downarrow \\ \text{Newton's method} \\ \text{solution after one iteration}}}{x'} + \dfrac{1}{2}\dfrac{f''(x)}{f'(x)}(x_r - x)^2 + \dots = x' + \underbrace{\dfrac{1}{2}\left(\dfrac{f''(x)}{f'(x)}\right)\varepsilon^2}_{\substack{\text{This is} \\ \varepsilon' = \text{error after} \\ \text{one iteration}}} + \dots$

Hence $\boxed{\varepsilon' = +\dfrac{1}{2}\dfrac{f''(x)}{f'(x)}\varepsilon^2}$

.) Newton's method converges quadratically. This is very fast:

$$\begin{cases} x_r = x + \varepsilon_0 \\ \varepsilon_1 = -\frac{1}{2}\frac{f''}{f'}\varepsilon_0^2 \simeq -c\,\varepsilon_0^2 \\ \varepsilon_2 \simeq -c\,\varepsilon_1^2 = c^3 \varepsilon_0^4 \\ \varepsilon_3 \simeq -c\,\varepsilon_2^2 = -c^7 \varepsilon_0^8 \\ \vdots \end{cases} \Rightarrow \boxed{|\varepsilon_N| \simeq \frac{(c\,\varepsilon_0)^{2^N}}{c}}$$

Exponential of an exponential!

iii) Set $x_r = x + \varepsilon_0$, $x_r = x_1 + c\,\varepsilon_0^2$. Hence:

$$x - x_1 + \varepsilon_0(1 - c\,\varepsilon_0) = 0 \; ; \quad x - x_1 + \varepsilon_0(1 - c\,\varepsilon_0) = 0$$

If ε_0 is small we set $\varepsilon_0(1 - c\,\varepsilon_0) \simeq \varepsilon_0$ and $\boxed{x_1 - x \simeq \varepsilon_0} \Rightarrow$ Convergence criterion