Thursday, 16 November 2023 08:34

Ex 2.7.12

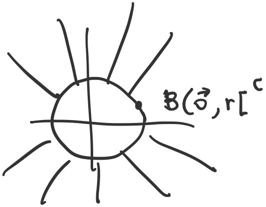
Is D open? closed? bded? compact?

Def: We say that D is compatt if D is closed and bounded.

Rmk: There is no relation between to be closed and to be bded:

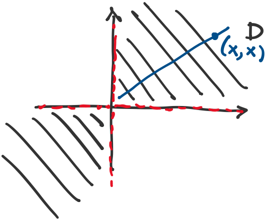
$D = \mathbb{R}^d$ closed, unbounded

$D = B(\vec{0}, r[$ is not closed, it is bded



i) $D = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$

$xy > 0 \iff x > 0,\ y > 0$ or $x < 0,\ y < 0$



D is open because $D = \{g > 0\}$ where $g(x,y) = xy \in \mathscr{C}(\mathbb{R}^2)$.

D is not closed, otherwise D would be $\phi, \mathbb{R}^2$.

D is unbdd : $(x,x) \in D \quad x > 0$

$$\|(x,x)\| = \sqrt{2x^2} = \sqrt{2}\,|x| = \underset{+\infty}{\underbrace{\sqrt{2}\,x}_{x\to+\infty}}$$

D is not compact because it is not closed nor bdd. ■

Rmk: To prove that D is bdd we need to find a universal constant M :

$$\|\vec{x}\| \leq M \qquad \forall \vec{x} \in D$$

or, equiv, bounds for each coords.

To prove that D is unbdd we should prove that

$$\nexists M : \|\vec{x}\| \leq M \quad \forall \vec{x} \in D$$

$$\Updownarrow$$

$$\forall M \quad \exists \vec{x}_M \in D : \|\vec{x}_M\| > M$$

In part, $M = n \in \mathbb{N}$

$$\exists\, \vec{x}_n \in D \;:\; \|\vec{x}_n\| > n$$

$$\Big\downarrow$$

$$+\infty$$

$$\Updownarrow$$

$$\boxed{\exists\, (\vec{x}_n) \subset D \;:\; \vec{x}_n \longrightarrow \infty_d .}$$

iii) $D = \{(x,y) \in \mathbb{R}^2 \;:\; \underbrace{|x| + |y| < 1}\}.$

Sol: $D = \{g(x,y) < 0\}$ where

$$g(x,y) = |x| + |y| - 1 \in \mathcal{C}(\mathbb{R}^2)$$

$\Rightarrow$ D is open

Since $D \neq \emptyset$ $((0,0) \in D)$, $D \neq \mathbb{R}^2$

$((1,1) \notin D)$ $\Rightarrow$ D cannot be closed.

D bded ?

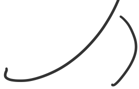
$$(x,y) \in D \iff \underbrace{\overset{+}{|x|} + \overset{+}{|y|}} < 1 \implies \boxed{\begin{matrix} |x| < 1 \\ |y| < 1 \end{matrix}}$$

$$\not\Leftarrow$$

$$\left[\quad |x| - |y| < 1 \;\not\Rightarrow\; \begin{matrix} |x| < 1 \\ |y| < 1 \end{matrix} \quad\right]$$

$$\Big(\quad \overset{\overset{0}{\wedge}}{a_1} + \overset{\overset{0}{\wedge}}{a_2} + \dots + \overset{\overset{0}{\wedge}}{a_N} \geq a_j$$

$$1 > |x| + |y| \geq |x|$$
$$\geq |y|$$



$\Rightarrow$ D bded.

D is not compact because it is not closed. ■

[Figure $|x| + |y| < 1$

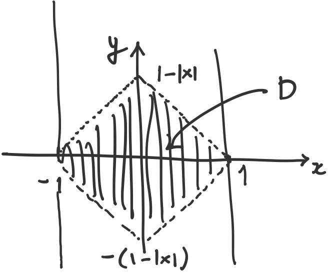
$|y| \textcircled{<} 1 - |x| \Leftrightarrow$ no y $a \leq 0$

$(|y| < a$ $-a < y < a$ $a > 0$

$a = 1 - |x| > 0 \Leftrightarrow |x| < 1 \Leftrightarrow -1 < x < 1$

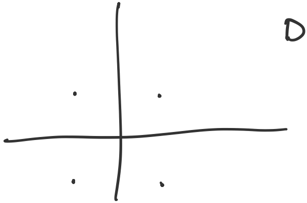
So

$$(x,y) \in D \Leftrightarrow \begin{cases} -(1-|x|) < y < 1-|x|, & -1 < x < 1 \\ \emptyset & x \leq -1 \vee x \geq 1 \end{cases}$$



Alt. arg: $|x| + |y| < 1$

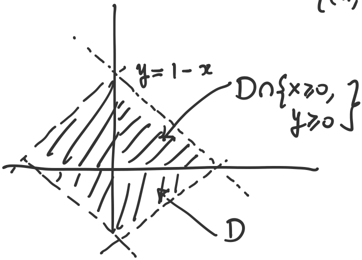
$(x,y)\in D \Rightarrow (-x,y),\ (x,-y),\ (-x,-y)\in D$



$$\underbrace{D\cap\{x\geqslant 0,\ y\geqslant 0\}}_{\text{first quarter}}$$

$$\|$$

$$\left\{(x,y):\ \substack{x\geqslant 0\\ y\geqslant 0},\ x+y<1\right\}$$



$y < 1-x$

$\lrcorner$

Do iv), v).

vi) $D = \{(x,y,z)\in\mathbb{R}^3 :\ z\geqslant x^2+y^2,\ x^2+z^2\leqslant 1\}$

Sol: $D = \{g_1(x,y,z)\leqslant 0,\ g_2(x,y,z)\leqslant 0\}$

where $\left.\begin{aligned} g_1 &= x^2+y^2-z \\ g_2 &= x^2+z^2-1 \end{aligned}\right\} \in \mathcal{C}(\mathbb{R}^3)$

$$D = \underbrace{\{g_1\leqslant 0\}}_{\text{closed}}\cap\underbrace{\{g_2\leqslant 0\}}_{\text{closed}} \Rightarrow D \text{ closed.}$$

Rmk: A set $\{g \geq 0\}$ can be open

$\{1 \geq 0\} = \mathbb{R}^d$ ■

Since $D \neq \emptyset$ $\big((0,0,0) \in D\big)$

$D \neq \mathbb{R}^3$ $\big((1,1,1) \notin D\big)$

$\Rightarrow$ D is not closed.

D bded ?

$$x^2 \leq z \qquad y^2 \leq z$$

$$(x,y,z) \in D \Leftrightarrow \begin{cases} \boxed{x^2 + y^2 \leq z} \\ \boxed{x^2 + z^2 \leq 1} \end{cases} \Rightarrow \boxed{\begin{matrix} x^2 \leq 1 \\ z^2 \leq 1 \end{matrix}} \Updownarrow -1 \leq z \leq 1$$

$$y^2 \leq \underbrace{z - x^2}_{\leq 1}$$

$$y^2 \leq 1$$

$$\Rightarrow \|(x,y,z)\|^2 = \underset{\leq 1}{x^2} + \underset{\leq 1}{y^2} + \underset{\leq 1}{z^2} \leq 3 \quad \forall (x,y,z) \in D.$$

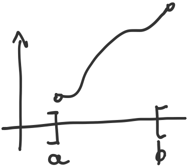
$\Rightarrow$ D is bded. ■

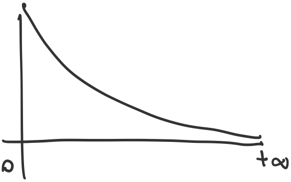
Do 2.7.12 vii), viii).

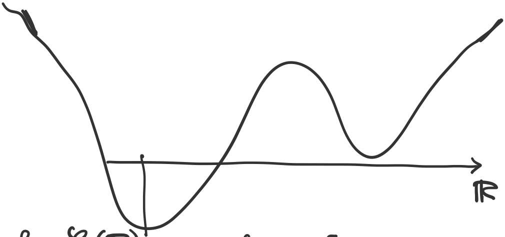
W. thm requires $f \in \mathcal{C}(D)$, D $\underbrace{\text{closed \& bded}}_{\text{compact}}$

What if D is not compact?

In general f has not min nor max







$f \in \mathcal{C}(\mathbb{R})$, $\displaystyle\lim_{x \to \pm\infty} f(x) = +\infty$

$\Rightarrow \exists$ global min for f on $\mathbb{R}$

Thm: $f \in \mathcal{C}(D)$, D closed and unbounded.

If

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty \qquad (*)$$

Then f has global min on D.

(and, of course, by $(*)$ $\nexists$ global max)

Similarly if

$$\lim_{\vec{x} \to \infty_d} f = -\infty$$

then f has global max on D.

Read and study sect 2.6

Diff. Calculus

Recall that if $f = f(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$

we call

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(provided the lim is finite)

If $\vec{f} = \vec{f}(\vec{x}) ; D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, can we

define

$$\vec{f}'(\vec{x}) = \lim_{\vec{h} \to \vec{0}} \frac{\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x})}{\boxed{\vec{h}}}$$

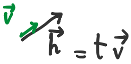
Pb: How can we divide by $\vec{h}$?

There're two sols to this pb.

Directional Derivative

Idea: $\vec{h} = t\vec{v}$

$\vec{h} \to \vec{0} \Leftarrow t \to 0$



$$\lim_{t \to 0} \frac{\vec{f}(\vec{x} + t\vec{v}) - \vec{f}(\vec{x})}{t}$$

Def: We say that $\vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ has directional derivative at point $\vec{x}$ along $\vec{v}$ if

$$\exists \underbrace{\lim_{t \to 0} \frac{\overset{\in \mathbb{R}^m}{\vec{f}(\vec{x} + t\vec{v})} - \overset{\in \mathbb{R}^m}{f(\vec{x})}}{t}}_{\in \mathbb{R}^m} =: D_{\vec{v}} \vec{f}(\vec{x})$$

Ex 3.1.2. Compute

$$D_{\underset{\vec{v}}{(1,1)}} f\overset{\vec{x}}{(0,0)} \qquad \text{for } f(x,y) = x \cos y.$$

Sol: By def

$$D_{(1,1)} f(0,0) = \lim_{t \to 0} \frac{f\big(\overset{\vec{x}}{(0,0)} + t\,\overset{\vec{v}}{(1,1)}\big) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(t,t) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{t} \cos t - \cancel{0}}{\cancel{t}}$$

$$= \lim_{t \to 0} \cos t = 1$$

$$\Rightarrow \quad \exists \; D_{(1,1)} f(0,0) = 1 \quad \blacksquare$$

Do Ex 3.6.1

iii) $D_{(1,1)} f(0,0)$ where $f(x,y) = \begin{cases} \dfrac{x^2 y}{|x|+y^2} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$

$$D_{\underbrace{(1,1)}_{\vec{v}}} f(\overset{\vec{x}}{\overbrace{0,0}}) = \lim_{t \to 0} \frac{f((0,0)+t(1,1)) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{\boxed{f(t,t)} - f(0,0)}{t}$$

$$= \lim_{\underbrace{t \to 0}_{(t,t) \neq \vec{0}}} \frac{\dfrac{t^2 \cdot \cancel{t}}{|t|+t^2} \; \cancel{-0}}{\cancel{t}}$$

$$= \lim_{t \to 0} \frac{t^2}{|t|+t^2} = \lim_{t \to 0} \frac{\overset{|t|^{\cancel{2}}}{\overset{\parallel}{t^2}}}{\cancel{|t|}\left(1 + \frac{|t|^{\cancel{2}}}{\cancel{|t|}}\right)}$$

$$= \lim_{t \to 0} \frac{|t| \to 0}{1_t \to 1} = 0. \quad \blacksquare$$

v) $D_{(-1,-2)} f(0,0)$ $\quad f(x,y) = \begin{cases} \dfrac{y(e^x - 1)}{\sqrt{x^2+y^2}} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$

$$D_{(-1,-2)}f(0,0) = \lim_{t\to 0} \frac{f\big((0,0)+t(-1,-2)\big) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(-t,-2t) - \overset{0}{\overset{\shortparallel}{f(0,0)}}}{t}$$

$$= \lim_{t\to 0} \frac{\dfrac{-2\cancel{t}\,(e^{-t}-1)}{\sqrt{t^2+4t^2}} \cancel{-0}}{\cancel{t}}$$

$$= \lim_{t\to 0} \boxed{-2}\, \frac{e^{-t}-1}{\sqrt{5t^2}} \quad = \sqrt{5}\,|t|$$

$$= -\frac{2}{\sqrt{5}} \lim_{t\to 0} \frac{e^{-t}-1 \to 0}{|t| \to 0}$$

$$= +\frac{2}{\sqrt{5}} \lim_{t\to 0} \underbrace{\boxed{\frac{e^{-t}-1}{-t}}}_{1} \cdot \overset{=\ \operatorname{sign} t}{\boxed{\frac{+t}{|t|}}}$$

$$t\to 0+ \qquad = \frac{2}{\sqrt{5}}\cdot 1\cdot 1 = \frac{2}{\sqrt{5}}$$

$$t\to 0- \qquad = \frac{2}{\sqrt{5}}\cdot 1\cdot (-1) = -\frac{2}{\sqrt{5}}$$

$$\Rightarrow \quad \nexists \; D_{(-1,-2)} f(0,0) \qquad \blacksquare$$

Unfortunately, the directional deriv. is not a satisfactory def of der. because of the content of the following example.

EX. Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This f verifies the following facts.

1. $\nexists \lim\limits_{(x,y)\to\vec{0}} f(x,y)$

2. $\exists \; D_{\vec{v}} f(\vec{0}\,) \quad \forall \vec{v} \in \mathbb{R}^2$

In part this f is not cont. at $\vec{0}$ even if it has all the directional derivatives.

Check of 1. and 2.

1. Adapt idea from a prev ex (Class 12)

2. $\vec{v} = (\alpha, \beta)$

$$D_{\vec{v}} f(\vec{0}) = \lim_{t \to 0} \frac{f\big(\overset{t\alpha,\, t\beta}{\overset{\parallel}{(0,0) + t(\alpha,\beta)}}\big) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(t\alpha, t\beta) - \overset{0}{\overset{\parallel}{f(0,0)}}}{t}$$

$t \neq 0$

$\vec{v} \neq \vec{0}$

(if $\vec{v} = \vec{0}$ $\quad D_{\vec{v}} f(\vec{x}) = \lim_{t \to 0} \frac{\overset{0}{\overset{\parallel}{f(\vec{x} + t\vec{0}) - f(\vec{x})}}}{t}$)

$= 0$

$$= \lim_{t \to 0} \frac{\dfrac{(t\alpha)^2 \cancel{t}\beta}{(t\alpha)^4 + (t\beta)^2}}{\cancel{t}}$$

$$= \lim_{t \to 0} \frac{\cancel{t^2}\, \alpha^2 \beta}{t^{\cancel{4}\,2} \alpha^4 + \cancel{t^2} \beta^2}$$

$$= \lim_{t \to 0} \frac{\alpha^2 \beta}{t^2 \alpha^4 + \beta^2} = \begin{cases} \dfrac{\alpha^2 \beta}{\beta^2} & \beta \neq 0 \\ \lim \dfrac{0}{t^2 \alpha^4} = 0 & \beta = 0 \end{cases}$$

$$D_{(\alpha,\beta)} f(\vec{0}) = \begin{cases} \dfrac{\alpha^2}{\beta} & \beta \neq 0 \\ 0 & \beta = 0 \end{cases}$$

∎