Wednesday, 15 November 2023 08:39

Thm (Cantor)

A set $D \subset \mathbb{R}^d$ is $\boxed{\text{closed}}$ $\Longleftrightarrow$

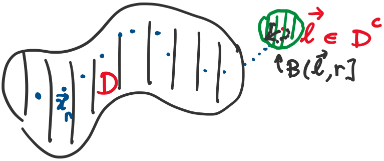
D contains all finite limits of sequences of vectors of D, that is

$$\boxed{\forall (\vec{x}_n) \subset D,\ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d \Rightarrow \vec{\ell} \in D}$$

Proof:

$\Rightarrow$ Hp: D closed

$\boxed{\text{Th}}$ $\forall (\vec{x}_n) \subset D,\ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R} \Rightarrow \vec{\ell} \in D$



Suppose $\boxed{\text{Th is false}}$,

$\exists\ \boxed{(\vec{x}_n) \subset D},\ \boxed{\vec{x}_n \longrightarrow \vec{\ell}} \in \mathbb{R},\ \vec{\ell} \notin D$

$\Rightarrow \vec{\ell} \in D^c$ open (by def since D is closed)

$$\Downarrow$$

$\exists\ B(\vec{\ell}, r] \subset D^c$

Since $\vec{x}_n \to \underset{\in \mathbb{R}^d}{\vec{\ell}} \Longleftrightarrow$

$$\forall \varepsilon > 0 \ \exists N : \underbrace{\| \vec{x}_n - \vec{\ell} \| \leq \varepsilon}_{\vec{x}_n \in B(\vec{\ell}, \varepsilon]} \quad \forall n \geq N$$

$$\text{If} \quad \varepsilon = r \Rightarrow \exists N : \vec{x}_n \in \underset{\underset{D^c}{\cap}}{B(\vec{\ell}, r]} \ \forall n \geq N$$

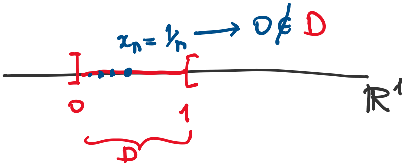
$$\Rightarrow \vec{x}_n \in D \quad \forall n \geq N$$

(impossible because $(\vec{x}_n) \subset D$)

$\Rightarrow$ Th is true.

$\Leftarrow$ Hp: $\boxed{\forall (x_n) \subset D, \ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d \Rightarrow \vec{\ell} \in D}$

Rmk: In gen, if $(\vec{x}_n) \subset D$, $\vec{x}_n \to \vec{\ell} \in \mathbb{R}^d$ the limit $\vec{\ell}$ does not belong to D



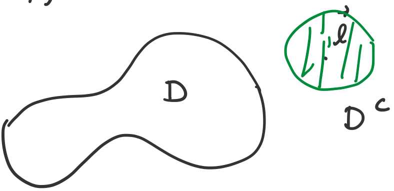
■

Th: D closed $\Leftrightarrow$ D^c open

If $D = \mathbb{R}^d \Rightarrow D^c = \emptyset \Rightarrow D^c$ open

$= \emptyset$

$D \neq \emptyset, \mathbb{R}^d$



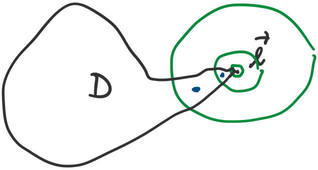
If $\boxed{\vec{\ell} \in D^c}$, we need to prove that

$$\vec{\ell} \in \operatorname{Int} D^c \Leftrightarrow$$

$$\boxed{\exists\ B(\vec{\ell}, r] \subset D^c}$$

If **false** $\Leftrightarrow$ $\forall\ \cancel{B(\vec{\ell}, r]} \subset D$

$$\forall r \quad B(\vec{\ell}, r] \cap D \neq \emptyset$$



$r = 1/n \qquad \exists \boxed{\vec{x}_n} \in B(\vec{\ell}, 1/n] \underline{\cap D}$

$$\Downarrow$$

$(\vec{x}_n) \subset D, \qquad \|\vec{x}_n - \vec{\ell}\| \leq 1/n \longrightarrow 0$

$$\Updownarrow$$

$$\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$$

$\overset{Hp}{\Rightarrow} \quad \vec{\ell} \in D$ but $\vec{\ell} \in D^c$

(imposs!)

$$\Rightarrow \quad \vec{\ell} \in \text{Int } D^c \qquad \forall \vec{\ell} \in D^c$$

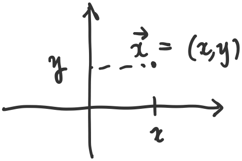
$$\Leftrightarrow \quad D^c \text{ open.}$$

$\blacksquare$

Cantor thm provides an useful (an simple) test to classify open/closed sets in certain standard cases.

Ex: $\mathbb{R}^2$, D = closed ball centred at $\vec{0}$ with radius r

$$D = B(\vec{0}, r] = \Big\{ \vec{x} : \underset{\substack{\| \\ \|\vec{x}\| = \sqrt{x^2+y^2}}}{\|\vec{x} - \vec{0}\|} \leq r \Big\}$$



$$D = \left\{ (x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} \leq r \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \boxed{x^2 + y^2 \leq r^2} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : g(x,y) \leq 0 \right\}$$

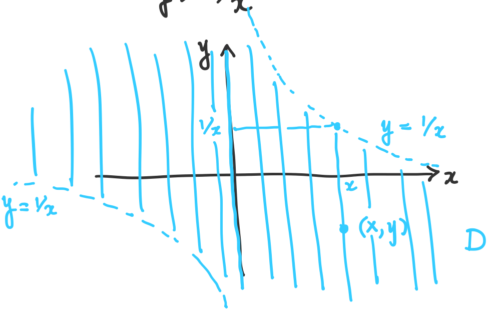
- $D = \{(x,y) \in \mathbb{R}^2 : \boxed{xy < 1}\}$

$g(x,y) < 0$

$\boxed{xy < 1}$

$$xy < 1 \iff \xcancel{\begin{cases} y < 1/x & x \neq 0 \\ 0 < 1 & x = 0 \end{cases}}$$

$$\iff \begin{cases} y < 1/x & x > 0 \\ 0 < 1 \quad \forall y & \underline{x = 0} \quad (0,y)\ \forall y \in \mathbb{R} \\ y > 1/x & x < 0 \end{cases}$$

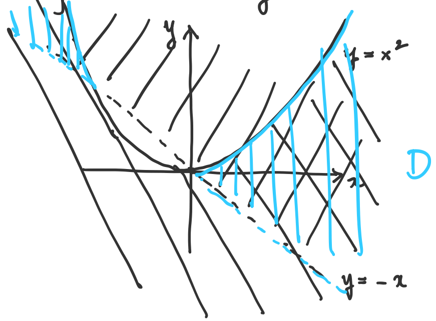


- $D = \{ x + y > 0 ,\ y \leq x^2 \}$

$$g_1(x,y) < 0 \quad , \quad g_2(x,y) \le 0$$

$$= \{x+y>0\} \cap \{y \le x^2\}$$

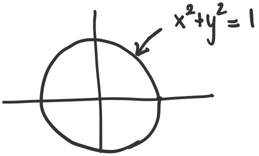
$x+y>0 \iff y > -x$



$y \le x^2$

$$g(x,y) = 0$$

. $D = \{x^2 + y^2 = 1\}$



In general, a natural way to define a set of pts of $\mathbb{R}^d$ is through a number of inequalities ($<$, $\le$, $>$, $\ge$) and/or equalities

constraint

$$D = \{ \boxed{g_1 < 0}, g_2 < 0, \dots, g_m < 0,$$
$$g_{m+1} \le 0, \dots \quad g_n \le 0,$$
$$g_{n+1} = 0, \dots \quad g_p = 0 \}$$

Corollary: If $g \in \mathcal{C}(\mathbb{R}^d)$ (continuous funct on $\mathbb{R}^d$) then:

- $\{ g \le 0 \}$, $\{ g = 0 \}$ are closed
- $\{ g < 0 \}$ are open

Proof: $\{ g \le 0 \}$ is closed.

We prove that

$$\underbrace{\forall (\vec{x}_n) \subset \{g \le 0\}}_{g(\vec{x}_n) \le 0 \ \forall n}, \ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d \Rightarrow \vec{\ell} \in \{g \le 0\}$$

$$\vec{x}_n \downarrow \vec{\ell} \quad \Rightarrow \quad \underset{\le 0}{g(\vec{x}_n)} \longrightarrow \underset{\le 0}{g(\vec{\ell})}$$

$$\le 0 \Rightarrow \le 0 \Rightarrow \vec{\ell} \in \{g \le 0\}$$

Ex: prove that $\{g = 0\}$ is closed,

$\{g < 0\}$ is open. ■

Exercise: True or False?

$$D \text{ closed} \Leftrightarrow \partial D \subset D$$

(prove this if true, provide a counterexample if false)

Weierstrass Thm

We remind that

Thm: If $f \in \mathcal{C}(\underset{\sim}{[a,b]})$, then f has both absolute/global min and max on $[a,b]$

that is:

$$\exists\, \underset{\uparrow}{x_{min}},\ \underset{\uparrow}{x_{max}} \in [a,b] : \underset{\nwarrow}{f(x_{min})} \le f(x) \le f(x_{max}) \quad \forall x \in [a,b]$$

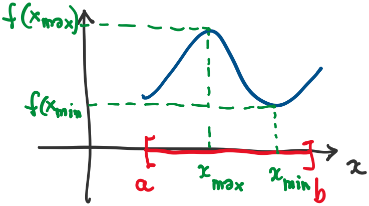
global min and max pts.

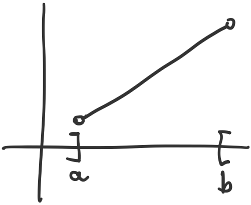
min max values of f on $[a,b]$

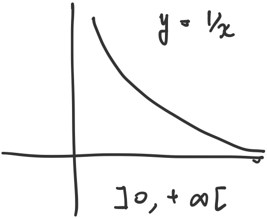
$$f(x_{min}) = \min\{f(x) : x \in [a,b]\} \equiv \min_{[a,b]} f$$

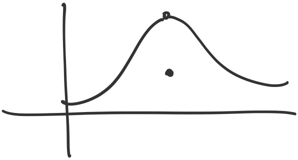
$$\equiv \min_{x \in [a,b]} f(x)$$

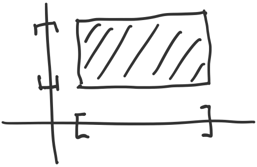
$$f(x_{max}) = \max\{f(x) : x \in [a,b]\} \equiv \max_{[a,b]} f$$

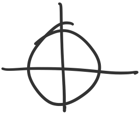












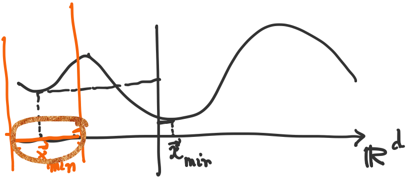
Let's consider now a function

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

Def: We say that $\vec{x}_{min} \in D$ is

- global (absolute) min point for f on D if

$$f(\vec{x}_{min}) \leqslant f(x) \quad \forall x \in D$$



- **local** (relative) **min pt** for f on D if

$$\exists\ B(\vec{x}_{min}, r] \ : \ f(\vec{x}_{min}) \leqslant f(\vec{x}) \ \forall \vec{x} \in D \cap B(\vec{x}_{min}, r]$$

Ex: Write similar defs for global max pt
local max pt.

Thm: If $f \in \mathscr{C}(D)$, D **closed** and **bounded**, then f has both global min and max on D,

$$\exists\ \vec{x}_{min}, \vec{x}_{max} \in D \ : \ f(\vec{x}_{min}) \leqslant f(\vec{x}) \leqslant f(\vec{x}_{max})$$
$$\forall \vec{x} \in D.$$

Def: A set $D \subset \mathbb{R}^d$ is said to be **bounded**

$\exists M : \quad \|\vec{x}\| \leq M \quad \forall \vec{x} \in D.$

Rmk: Since

$$\|\vec{x}\| = \sqrt{\sum_{j=1}^{d} x_j^2} \leq M$$

$$\begin{array}{c} \| \\ \sqrt{x_1^2 + x_2^2 + \dots + x_d^2} \\ \geq \end{array}$$

$$|x_j| = \sqrt{x_j^2} \leq M$$

$\Rightarrow$ D is bounded $\Leftrightarrow$ all the coords of pts of D are bded. ■

Ex 1: $D = \{(x,y) : \underbrace{x^2 + 2y^2 \leq 1}\}$

Is D closed & bded ?

Sol:

- $D = \{ g(x,y) \leq 0 \}$ where

$$g(x,y) = x^2 + 2y^2 - 1 \in \mathscr{C}(\mathbb{R}^2)$$

$\Rightarrow$ $\underline{D \text{ is closed}}$ (by Cantor corollary)

- Is D bded ?

$(x,y) \in D \qquad x^2 + 2y^2 \leq 1$

$$\|(x,y)\|^2 = x^2 + y^2 \leq x^2 + 2y^2 \leq 1$$

$$\Rightarrow \quad \|(x,y)\|^2 \leq 1$$

$$\Rightarrow \quad \|(x,y)\| \leq \underset{\substack{\shortparallel \\ \ddot{M}}}{1} \qquad \forall (x,y) \in D$$

(alt arg:

$(x,y) \in D \iff x^2 + 2y^2 \leq 1$

$$\Rightarrow \begin{cases} x^2 \leq 1 \\ 2y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} |x| \leq 1 \\ |y| \leq 1/\sqrt{2} \end{cases}$$

Coords are bded

$\Downarrow$

D is bded)

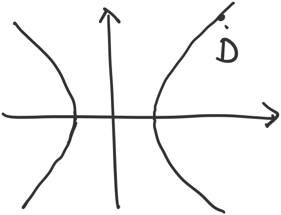
$\Rightarrow$ D is closed & bded. ■

Ex: $D = \{ x^2 - y^2 = 1 \}$ is D closed & bded?

Sol: $D = \{ g(x,y) = 0 \} \qquad g = x^2 - y^2 - 1 \in \mathcal{C}(\mathbb{R}^2)$

$\Rightarrow$ D is closed by a gen fact.

. D bded ?



$(x,y) \in D \iff x^2 - y^2 = 1 \iff y^2 = x^2 - 1$

$(x, \pm\sqrt{x^2-1}) \in D \quad \forall |x| \geq 1$

$\|(\qquad\qquad)\|^2 = x^2 + (x^2-1)$

$= 2x^2 - 1 \xrightarrow[x \to \pm\infty]{} +\infty$

$\Rightarrow$ D is unbded. ▨

Do Ex 2.7.12.