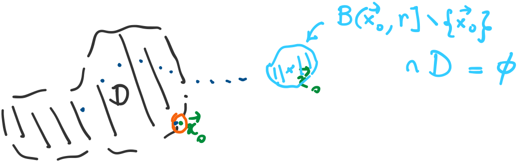
Monday, 13 November 2023 08:37

Prop: $D \subset \mathbb{R}^d$, $\underline{\vec{x}_0 \in \mathbb{R}^d}$

$\vec{x}_0$ is acc pt for D

$$\Updownarrow$$

$$\forall r \quad \underline{B(\vec{x}_0, r]} \setminus \{\vec{x}_0\} \cap D \neq \phi .$$



Proof:

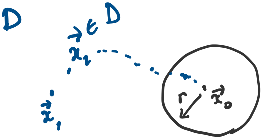
Hp: $\boxed{\vec{x}_0 \text{ is acc. pt for } D}$

Th: $\forall r \quad \underline{B(\vec{x}_0, r]} \setminus \{\vec{x}_0\} \cap D \neq \phi .$

$$Hp \overset{\text{by def}}{\iff} \exists \boxed{(\vec{x}_n)_{n \geq 1} \subset D}, \boxed{\vec{x}_n \neq \vec{x}_0}, \vec{x}_n \to \vec{x}_0$$

$$\Updownarrow \text{ by def}$$

$$\forall \underset{\smile}{\varepsilon > 0} \ \exists N : \boxed{\|\vec{x}_n - \vec{x}_0\| \leq \varepsilon} \quad \forall n \geq N$$



$$\varepsilon = r : \quad \exists N : \|\vec{x}_n - \vec{x}_0\| \leq r \quad \forall n \geq N$$

$$\Updownarrow$$

and since $\boxed{\begin{matrix}\vec{x}_n \in B(\vec{x}_0, r] \\ \vec{x}_n \neq \vec{x}_0 \quad \forall n\end{matrix}}$ $n \geq N$

$$\Rightarrow \underset{\boxed{D}\cap}{} \vec{x}_n \in \boxed{B(\vec{x}_0, r] \setminus \{\vec{x}_0\}}$$

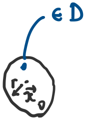
$$\Rightarrow \quad \vec{x}_n \in B(\vec{x}_0, r] \setminus \{\vec{x}_0\} \cap D \quad \forall n \geq N.$$

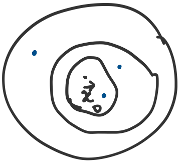
Th is true.

Vice versa

Hp: $\forall r>0, \ \boxed{B(\vec{x}_0, r] \setminus \{\vec{x}_0\}} \cap D \neq \phi \leftarrow$

Th: $\boxed{\vec{x}_0 \text{ is acc pt. for } D.}$





Take $r = 1/n \underset{\text{by the hp}}{\Rightarrow} \forall n \geq 1$ $B(\vec{x}_0, 1/n] \setminus \{\vec{x}_0\} \cap D \overset{\phi}{\neq}$

$$\Rightarrow \exists \vec{x}_n \in B(\vec{x}_0, \overset{3}{\boxed{1/n}}] \setminus \{\vec{x}_0\} \cap D$$

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$$\Rightarrow \quad \underset{1}{(\vec{x}_n) \subset D}, \quad \underset{2}{\vec{x}_n \overset{1}{\neq} \vec{x}_0}, \quad \text{and since}$$

$$0 \leq \|\vec{x}_n - \vec{x}_0\| \leq 1/n$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$0 \qquad 0 \qquad 0$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow \vec{x}_0 . \qquad \blacksquare$$

Recall that

∞_d is acc pt for $D \subset \mathbb{R}^d$

$\Leftrightarrow$ (def) $\exists\ (\vec{x}_n) \subset D$, $\cancel{(\vec{x}_n \neq \infty_d)}$, $\vec{x}_n \to \infty_d$

$$\Updownarrow$$

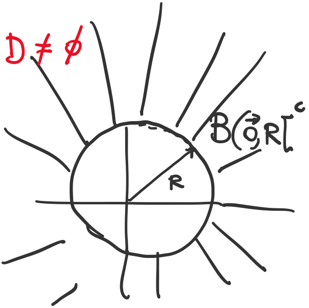
$\|\vec{x}_n\| \to +\infty$.

Proof: $D \subset \mathbb{R}^d$. Then

∞_d is acc pt for D

$$\Updownarrow$$

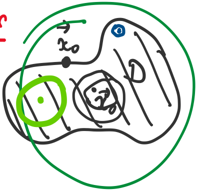
$\forall R > 0 \qquad B(\vec{0}, R[^c \cap D \neq \emptyset$



Do the proof of this prop as exercise.

Def: $D \subset \mathbb{R}^d$, $\vec{x}_0 \in \mathbb{R}^d$. We say that

$\vec{x}_0$ is in the interior of D if



$$\exists\, r>0:\ B(\vec{x}_0, r] \subset D$$

(clearly this def implies $\vec{x}_0 \in D$).

The set of all interior pts of D is called Int D.

$$\text{Int } D = \{\vec{x}_0 \in D : \exists\, B(\vec{x}_0, r] \subset D\}$$

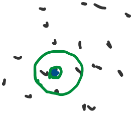
Rmks:

- Int $D \subset D$.
- Int D can be empty even if $D \neq \emptyset$

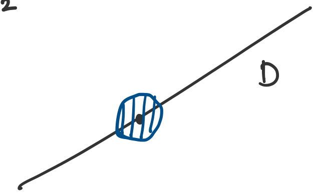
$D = \{\vec{x}_0\}$

$\nexists\, B(\vec{x}_0, r] \subset D$.

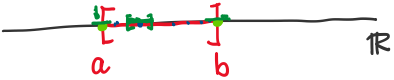
D = finite set



D = line in $\mathbb{R}^2$



. Int D can be $\neq \emptyset$ and $\subsetneq D$.



$$\mathrm{Int}\ [a,b] = \,]a,b[$$
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Exercise: In $\mathbb{R}^d$

$$\mathrm{Int}\ B(\vec{x}_0, r] = B(\vec{x}_0, r[\,.$$

Def: We say that $D \subset \mathbb{R}^d$ is open if

$$\mathrm{Int}\ D = D.$$

Examples:

- Since $\text{Int }]a,b[\; = \;]a,b[$

  $\Rightarrow \;]a,b[$ is an open set of $\mathbb{R}$.

- $\text{Int }]a,+\infty[\; = \;]a,+\infty[\; \Rightarrow \;]a,+\infty[$ is open

- $D = \phi, \; \mathbb{R}^d$ are open sets.

- $D = B(\vec{x}_0, r[$ is open (check this for exercise)

Prop: If A, B are open sets of $\mathbb{R}^d$

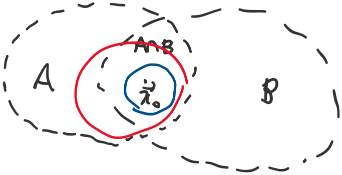
$\Rightarrow \; \boxed{A \cap B}, \; A \cup B$ are open.

Do the proof as ex.

Γ If one or both are empty or $\mathbb{R}^d$ concl is trivial

The true proof is for $\phi \neq A, B \neq \mathbb{R}^d$

For $A \cap B$ (assuming $A \cap B \neq \phi$ otherwise is trivial)



Extensions:

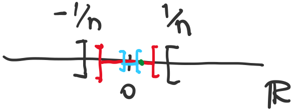
if $A_1, \dots, A_N$ are open sets of $\mathbb{R}^d$

$$\bigcap_{j=1}^{N} A_j \ , \quad \bigcup_{j=1}^{N} A_j$$ are both open.

(prove as exercise)

What if we have infinitely many sets :

- in general, the intersection is not open



$A_n =]-1/n, 1/n[$

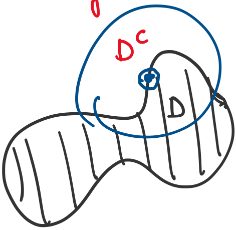
open $\forall n$.

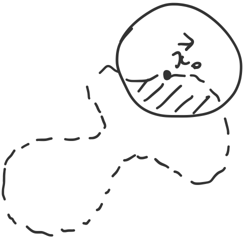
$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$ ← is not open

Prop: If $(A_n)_{n\in\mathbb{N}}$ is a family of open sets

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \text{ is open.}$$

Boundary / Fronteer of D





Def: Let $D \subset \mathbb{R}^d$. A point $\vec{x}_0 \in \mathbb{R}^d$ is a boundary pt for D if

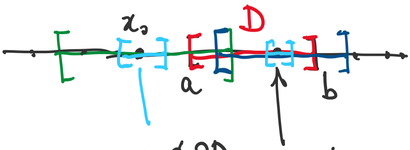
$$\forall r \quad \cancel{B(\vec{x}_0, r] \cap D \cap D^c \neq \emptyset}$$

$$\forall r \qquad \begin{aligned} &B(\vec{x}_0, r] \cap D \neq \emptyset \\ &B(\vec{x}_0, r] \cap D^c \neq \emptyset \end{aligned}$$

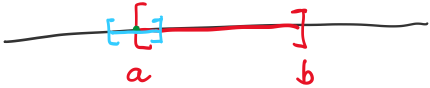
The set of all boundary pts of D is denoted by $\underset{\uparrow}{\partial} D$, and it is called boundary of D.

Examples:

- In general $\partial D \not\subset D$



$x_0 \notin \partial D$ $\quad$ $x_0 \notin \partial D$



$a \in \partial D$ $\quad$ $b \in \partial D$

$\partial\,[a,b] = \{a,b\}$ $\quad$ (here $\partial D \subset D$)

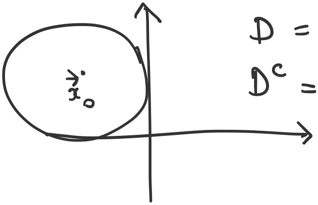
$\partial\,]a,b[= \{a,b\}$ $\quad$ (here $\underset{\underset{D^c}{\cap}}{\partial D} \not\subset D$)

$\partial\,[a,b[= \{a,b\}$

(here $\partial D \not\subset D, D^c$)

· $\partial D = \phi$?

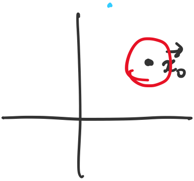
$D = \mathbb{R}^d, \phi$ $\quad$ $\partial D = \phi$



$D = \mathbb{R}^2$

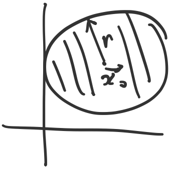
$D^c = \phi$

$B(\vec{x}_0, r] \cap D^c = \phi$
$\forall r$



- $\partial \{\vec{x}_0\} = \{\vec{x}_0\}$.

Exercise : $\partial B(\vec{x}_0, r] = \{\vec{x} \in \mathbb{R}^d : \|\vec{x} - \vec{x}_0\| = r\}$



Def: A set $D \subset \mathbb{R}^d$ is said to be closed if A^c is open

Rmks:

- A set can be both open and closed

$D = \phi$ open, $D^c = \mathbb{R}^d$ open

$\Downarrow$

D is closed.

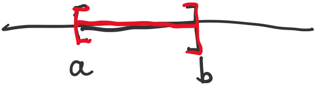
$D = \mathbb{R}^d$ open, $D^c = \emptyset$ open

$$\Downarrow$$

D is closed

It can be proved that $\emptyset$ and $\mathbb{R}^d$ are the unique subsets of $\mathbb{R}^d$ to be both open and closed.

- In gen a set is neither open nor closed.



$[a,b]$ is closed $\left([a,b]^c = \underbrace{\underbrace{]-\infty, a[}_{\text{open}} \cup \underbrace{]b,+\infty[}_{\text{open}}}_{\text{open}} \right)$

but it is not open

$$\left(\underbrace{\text{Int } [a,b]} =]a,b[\neq [a,b] \right)$$

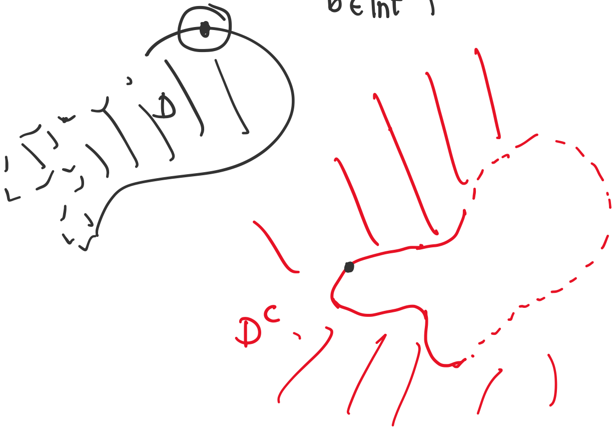
$]a,b[$ is open but not closed

$$\left( ]a,b[^c =]-\infty, a] \cup [b, +\infty[\right)$$

$a, b \notin \text{Int}$ ↑

$[a,b[$ this is not open $\left(a \notin \text{Int}\,[a,b[\right)$

and it is also not closed

$$\left([a,b[^c = \underbrace{]-\infty, a[\cup [b,+\infty[}_{b \in \text{Int}} \right)$$