Monday, 6 November 2023 08:42

Focus on

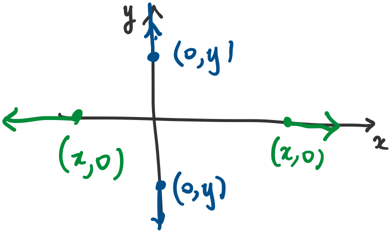
$$\lim_{\vec{x} \to \infty_d} f(\vec{x})$$

**Example** Discuss existence and value of

$$\lim_{(x,y) \to \infty_2} \underbrace{(x^2 + y^2 + 3xy)}_{f(x,y)}$$

Sol: $f$ is def'd on $D = \mathbb{R}^2$

( $\infty_2$ is an acc pt for $D$ )



$$(x,0) \to \infty_2 \iff \|(x,0)\| \to +\infty$$

$$\| \quad\quad$$

$$|x| \to +\infty$$

$$\text{“}\Leftrightarrow\text{”} \quad \begin{array}{l} x \to +\infty \\ x \to -\infty \end{array}$$

$$f(x,0) = x^2 \longrightarrow +\infty \qquad |x| \longrightarrow +\infty$$

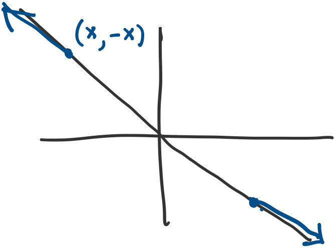
$$f(0,y) = y^2 \longrightarrow +\infty \qquad |y| \longrightarrow +\infty$$

$$\Updownarrow$$

$$(0,y) \to \infty_2 .$$

$$f(x,y) = x^2 + y^2 + 3xy$$

$$y = -x$$



$$(x,-x) \longrightarrow \infty_2 \iff \|(x,-x)\| \longrightarrow +\infty$$

$$\|(x,-x)\| = \sqrt{x^2+(-x)^2} = \sqrt{2}\,|x|$$

$$\iff \boxed{|x| \longrightarrow +\infty}$$

$$f(x,-x) = x^2 + (-x)^2 + 3x\cdot(-x)$$

$$= 2x^2 - 3x^2 = -x^2 \longrightarrow -\infty$$

$$\Rightarrow \quad \nexists \lim_{(x,y)\to\infty_2} f. \qquad \blacksquare$$

Ex:

$$\lim_{(x,y)\to\infty_2} \overbrace{\left(x^2+y^2+xy\right)}^{f}$$

Sol: Here f is dedf on $D=\mathbb{R}^2$

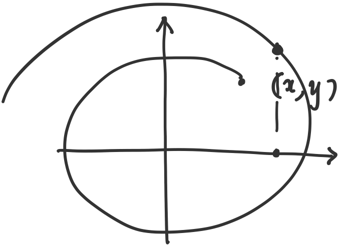
Sects:

$$f(x,0) = x^2 \longrightarrow +\infty \qquad |x| \longrightarrow +\infty$$

$$\Updownarrow$$

$$(x,0) \to \infty_2 .$$

$$f(0,y) = y^2 \longrightarrow +\infty \qquad |y| \to +\infty .$$



$$(x,y) \longrightarrow \infty_2$$

$$\Updownarrow \qquad \text{FALSE}$$

$$|x| \to +\infty$$

$$|y| \longrightarrow +\infty$$

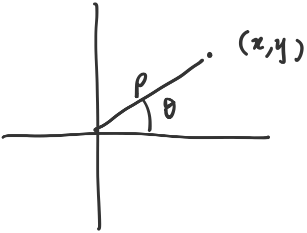
$$(x_n,y_n) = (n\cos n, n\sin n) \longrightarrow \infty_2$$

( because $\|(x_n,y_n)\| = \sqrt{n^2} = n \xrightarrow[n\to+\infty]{} +\infty$)

but $x_n = n\cos n$

$$|x_n| = \underset{\downarrow}{n}\,\underbrace{|\cos n|}_{} \qquad \nexists$$

$$\overset{+\infty}{\nexists \lim_{n} y_n} \quad .$$



$$(x,y) \to \infty_2 \iff \|(x,y)\| \to +\infty$$

$$\iff \boxed{\rho \to +\infty} \qquad \vartheta \in [0,2\pi]$$

$$f(x,y) \overset{\substack{x = \rho\cos\theta \\ y = \rho\sin\theta}}{=} \underbrace{\rho^2}_{x^2+y^2} + \underbrace{\rho^2\cos\theta\sin\theta}_{xy}$$

$$f = \underset{\substack{\swarrow \\ +\infty}}{\rho^2} \underbrace{\left(1 + \tfrac{1}{2}\boxed{2\cos\theta\sin\theta}\right)}_{\sin(2\theta)}$$

$$= \rho^2 \underbrace{\left(1 + \frac{1}{2}\underset{-1\le \quad \le 1}{\sin(2\theta)}\right)}$$

$$1-\tfrac{1}{2} \leq \qquad\qquad \leq 1+\tfrac{1}{2}$$

$$\boxed{\underset{\displaystyle\downarrow\atop\displaystyle +\infty}{\tfrac{1}{2}\rho^2} \leq f \leq \underset{\displaystyle\downarrow\atop\displaystyle +\infty}{\tfrac{3}{2}\rho^2}} \qquad \rho \to +\infty$$

$$\Rightarrow \quad \exists \lim_{(x,y)\to\infty_2} f = +\infty. \qquad \blacksquare$$

## Ex 2.2.11

$$\lim_{(x,y)\to\infty_2} \underbrace{\left(x^4+y^4-xy\right)}_{f}$$

Sol: Here $f$ is defd on $D=\mathbb{R}^2$.

$$f(x,0) = x^4 \longrightarrow +\infty \qquad (x,0)\to\infty_2$$

$$f \underset{y\,=\,\rho\sin\theta}{\overset{x\,=\,\rho\cos\theta}{=}}$$

$$= \rho^4\cos^4\theta + \rho^4\sin^4\theta - \rho^2\cos\theta\sin\theta$$

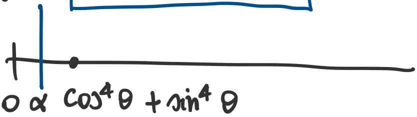
$$= \boxed{\rho^4}\underbrace{\overbrace{\left(\cos^4\theta+\sin^4\theta\right)}^{\geq \alpha}}_{>0} - \tfrac{1}{2}\boxed{\rho^2}\underbrace{\sin(2\theta)}$$

$$-\tfrac{1}{2}\rho^2 \qquad 0\leq\theta\leq 2\pi.$$

$$(x,y)\to\infty_2 \iff \rho\to+\infty$$

Pb: we know that

$$g(\theta) = \boxed{\cos^4\theta + \sin^4\theta} > 0 \qquad \forall\, \theta \in [0, 2\pi]$$



Call $g(\theta) := \cos^4\theta + \sin^4\theta.$

$$g \in \mathcal{C}([0,2\pi]) \Rightarrow (\text{W. thm})\ \exists \min_{[0,2\pi]} g = g(\theta^*)$$

$$\Rightarrow \quad g(\theta) \geqslant \underset{\alpha > 0}{\underset{\shortparallel\!:}{g(\theta^*)}} \qquad \forall\, \theta \in [0,2\pi]$$

$$\Rightarrow \quad \cos^4\theta + \sin^4\theta \geqslant \alpha > 0 \qquad \forall\, \theta \in [0,2\pi]$$

$$\Rightarrow \quad f \geqslant \boxed{\rho^4 \cdot \alpha - \frac{1}{2}\rho^2}$$

$$f = \rho^4 \big( \underset{\leq 1}{\cos^4\theta} + \underset{\leq 1}{\sin^4\theta} \big) - \underset{\leqslant \frac{1}{2}\rho^2}{\frac{1}{2}\rho^2 \sin 2\theta}$$

$$\leqslant 2\rho^4 + \frac{1}{2}\rho^2$$

$$\Rightarrow \underset{\overset{\vee}{0}}{\alpha}\rho^4 - \frac{1}{2}\rho^2 \leqslant f \leqslant 2\rho^4 + \frac{1}{2}\rho^2$$

$$\begin{tikzcd}
\alpha\rho^4 - \frac{1}{2}\rho^2 \arrow[d] & & 2\rho^4 + \frac{1}{2}\rho^2 \arrow[d] & \\
+\infty & & +\infty & \rho \to +\infty
\end{tikzcd}$$

$$\Rightarrow \lim_{(x,y)\to\infty_2} f = +\infty. \qquad \blacksquare$$

Ex 2.2.12

Compute $\displaystyle \lim_{(x,y,z)\to\infty_3} \underbrace{\left((x^2+y^2+z^2)^2 - xyz\right)}_{f}.$

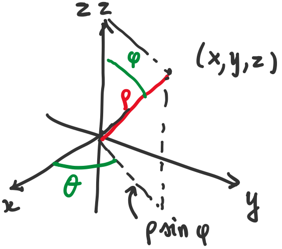
Sol: Here $f$ is def'd on $D = \mathbb{R}^3$.

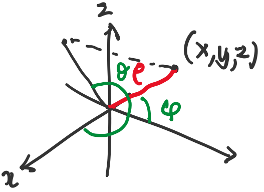
$$f(x,0,0) = x^4 \longrightarrow +\infty \quad \text{if}$$

$$(x,0,0) \to \infty_3$$

$$\Updownarrow$$

$$|x| \longrightarrow +\infty.$$





$$\begin{cases} x = \rho \sin\varphi \cdot \cos\theta \\ y = \rho \sin\varphi \cdot \sin\theta \\ z = \rho \cos\varphi \end{cases} \qquad \rho = \|(x,y,z)\| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \quad f = (\rho^2)^2 \ominus \rho^3 \boxed{\sin^2\varphi \cos\varphi \cos\theta \sin\theta}$$

$$-1 \le \qquad \le 1$$

$$f = \rho^4 - \rho^3 \underline{g(\varphi,\theta)}$$

$$\underset{\substack{\downarrow \\ +\infty}}{\rho^4 - \rho^3} \le f \le \underset{\substack{\downarrow \\ +\infty}}{\rho^4 + \rho^3} \qquad \rho \to +\infty$$

$$\Rightarrow \quad \exists \lim_{\infty_3} f = +\infty . \qquad \blacksquare$$

Do Ex 2.7.9.

## Continuity

**Def:** Let $\vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. We say that $\vec{f}$ is cont at $\vec{x}_0 \in D \cap \operatorname{Acc} D$

if

$$\lim_{\vec{x}\to\vec{x}_0} \vec{f}(\vec{x}) = \boxed{\vec{f}(\vec{x}_0)}.$$

Some quick rmks on this def:

**Prop:** If $\vec{f} : D \subset \mathbb{R}^d \longrightarrow \boxed{\mathbb{R}^m}$,

$$\vec{f}(\vec{x}) = \left(f_1(\vec{x}), f_2(\vec{x}), \dots\dots f_m(\vec{x})\right)$$

where $f_j : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ are the components of $\vec{f}$

⌈

$$\vec{f}(x,y) = \left(\overbrace{x+y}^{f_1(x,y)}, \overbrace{xy}^{f_2(x,y)}, \overbrace{x/y}^{f_3(x,y)}\right)$$

$$\vec{f} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

⌋

then

$\vec{f}$ is cont at $\vec{x}_0 \iff f_j$ is cont at $\vec{x}_0$ $\forall j = 1 \dots m$

⌈Do (or try to do) the proof recalling that $(\vec{y}_k) \subset \mathbb{R}^m$

$$\vec{y}_k \longrightarrow \vec{\ell} \in \mathbb{R}^m$$

$$\Updownarrow$$

$$y_{k,j} \longrightarrow \ell_j$$

$$\forall j = 1, \dots, m$$

use this with $y_j = f_j(\vec{x}) \dots\dots$

$\vec{\ell} = \vec{f}(\vec{x}_0) \in \mathbb{R}^m$. $\lrcorner$

**Rmk 2:** Some examples of cont funct numerical valued.

- $f(x_1, \dots, x_d) =$ polynomial in $(x_1 \text{—} x_d)$

$$= \sum c \, x_1^{j_1} x_2^{j_2} \cdots x_d^{j_d}$$

$$j_1 \text{—} j_d \in \mathbb{N}$$

Polynomials are cont functs on $\mathbb{R}^d$.

- The norm is a cont funct on $\mathbb{R}^d$

$$f(\vec{x}) := \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$$

$$= \sqrt{\underbrace{p(x_1 \text{—} x_d)}_{\text{polyn.}}} = \sqrt{p(\vec{x})}$$

so if $\vec{x} \to \vec{x}_0 \Rightarrow p(\vec{x}) \to p(\vec{x}_0)$

$$\overset{\text{cont of } \sqrt{\ }}{\Rightarrow} \quad \sqrt{p(\vec{x})} \to \sqrt{p(\vec{x}_0)}$$

$$\begin{array}{c} \| \\ f(\vec{x}) \to f(\vec{x}_0). \end{array}$$

- If $f(\vec{x})$ is cont and $\varphi = \varphi(u)$ $\varphi : \mathbb{R} \to \mathbb{R}$ is cont $\Rightarrow$

$$\varphi(f(\vec{x})) \text{ is cont.}$$

So, for ex:

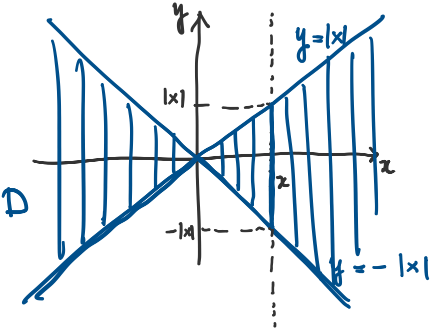
$$f(x,y) := \sqrt{x^2 - y^2}$$

$f$ is defd on $D = \{(x,y) : x^2 - y^2 \geq 0\}$

$$= \{(x,y) : y^2 \leqslant x^2\}$$

$$\Updownarrow$$

$$|y| \leqslant |x|$$



$$= \{(x,y) : -|x| \leqslant \underline{y \boxed{\leqslant} |x|\}}$$

$f = \sqrt{p(x,y)}$ where $p(x,y) = x^2 - y^2$

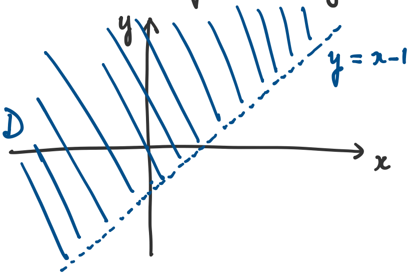
$= \varphi(p(x,y))$ " $\varphi(u) = \sqrt{u}$

$\Rightarrow$ $f$ is cont on $D$. ■

- $f(x,y) = \log(\boxed{1-x+y})$.

This $f$ is defd on $D = \{(x,y) : \underline{1-x+y>0}\}$

$= \{(x,y) : y > x-1\}$



$f(x,y) = \varphi(p(x,y))$

$p(x,y) = 1-x+y$

$\varphi(u) = \log u$

$\Rightarrow$ $f$ is cont on $D$. ■

- If $A$ is an $\underbrace{m}_{\text{lines}} \times \underbrace{d}_{\text{cols}}$ matrix, the function

$$\vec{f}(\vec{x}) := A\vec{x} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1d} \\ a_{21} & a_{22} -- & a_{2d} \\ \vdots & & \\ a_{m1} & ---- & a_{md} \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_d \end{bmatrix}$$

(linear funct)

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{md}x_d \end{bmatrix}$$

$$= \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

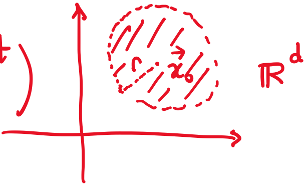
is cont. on $\mathbb{R}^d$. $\blacksquare$

## Basic Topological Defs

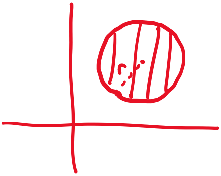
**Def:** (Open and Closed balls)

$$B(\vec{x}_0, r[ := \{ \vec{x} \in \mathbb{R}^d : \|\vec{x} - \vec{x}_0\| < r \}$$

(open ball centred at $\vec{x}_0$ with radius $r$)

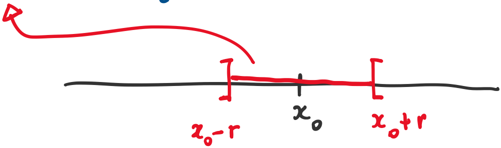


$$B(\vec{x}_0, r] := \{\vec{x}\in\mathbb{R}^d : \|\vec{x}-\vec{x}_0\| \leq r\}$$



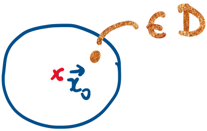
**Rmk:** $d=1$

$$B(x_0, r[ = \{x\in\mathbb{R} : |x-x_0|<r\}$$



**Prop:** $D\subset\mathbb{R}^d$.

- $\vec{x}_0\in\mathbb{R}^d$ is acc pt for D $\Leftrightarrow$

$$\forall r\ \left(B(\vec{x}_0, r]\setminus\{\vec{x}_0\}\right)\cap D\neq\emptyset$$