COMPUTABILITY (06/11/2023)

* DIAGONALISATION

idea : $x_i \quad i \in I$

$$x_0 \quad x_1 \quad x_2 \quad x_3 \quad \ldots\ldots \quad \underset{\substack{\uparrow \\ \text{position } k \text{ of } x_i}}{x_i}$$

aim : build x s.t.

$x \neq x_i$ — x differs from x_i at "position" i

Cantor $\forall X$ set

$$|X| < |2^X| \qquad 2^X = \{Y \mid Y \subseteq X\}$$

if X is finite $\quad X = \{0,1\} \quad 2^X = \{\emptyset, \{0\}, \{1\}, \{1,2\}\}$

$$|X| = 2 < |2^X| = 2^{|X|} = 2^2 = 4$$

Example : $|\mathbb{N}| < |2^{\mathbb{N}}|$

proof

assume $|\mathbb{N}| \geq |2^{\mathbb{N}}|$ i.e. $2^{\mathbb{N}}$ countable ($\underset{\text{surjective}}{\mathbb{N} \to 2^{\mathbb{N}}}$)

$2^{\mathbb{N}}$

	X_0	X_1	X_2	....
0	~~YES~~ NO	NO	NO	
1	NO	~~NO~~ YES	NO	
2	YES	NO	~~YES~~ NO	
3	NO	YES	YES	
⋮	NO	⋮	⋮	
	⋮			

$X_0 = \{0, 2\}$

$$D = \{i \mid i \notin X_i\} \subseteq \mathbb{N}$$

$$\Rightarrow \exists k \in \mathbb{N} \text{ s.t. } D = X_k$$

problem: $\quad K \in D$?

- yes: $K \in D \Rightarrow K \notin X_K = D$ $\quad$ contradiction
- no: $K \notin D \Rightarrow K \in X_K = D$ $\quad$ "

$\Rightarrow \quad 2^{\mathbb{N}}$ is not countable $\qquad |\mathbb{N}| < |2^{\mathbb{N}}|$

$\square$

EXERCISE: $\quad \mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$

$|\mathcal{F}| > |\mathbb{N}|$

(1st possibility)

$\mathcal{F}_2 = \{ f \in \mathcal{F} \mid f : \mathbb{N} \to \mathbb{N} \text{ total},\ img(f) \subseteq \{0,1\} \} \subseteq \mathcal{F}$

bijection $\mathcal{F}_2 \to 2^{\mathbb{N}}$

$f \longmapsto \{ n \mid f(n) = 1 \}$

$$\begin{array}{ccc} |\mathcal{F}_2| & = & |2^{\mathbb{N}}| \\ \wedge| & & \vee \\ |\mathcal{F}| & & |\mathbb{N}| \end{array}$$

$\mathcal{F}_2 \subseteq \mathcal{F} \rightsquigarrow$

$\mathcal{F}_2 \to \mathcal{F}$ injective

$f \mapsto f$

(2nd possibility) $\qquad |\mathcal{F}| > |\mathbb{N}|$

consider an enumeration of elements in $\mathcal{F}$

$$\begin{array}{c|cccc} & f_0 & f_1 & f_2 & f_3 \ \cdots \\ 0 & f_0(0) & f_1(0) & f_2(0) & \cdots \\ 1 & f_0(1) & f_1(1) & f_2(1) & \cdots \\ 2 & f_0(2) & f_1(2) & f_2(2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

$+1$ $\quad \uparrow \to 0$

$f : \mathbb{N} \to \mathbb{N}$

$$f(n) = \begin{cases} f_n(n) + 1 & f_n(n) \downarrow \\ 0 & f_n(n) \uparrow \end{cases}$$

we have $f \neq f_n \ \forall n$ since $f(n) \neq f_n(n)$ by construction

Hence there is no enumeration of all the functions in $\mathcal{F}$

$\Rightarrow |\mathcal{F}| > |\mathbb{N}|$

$\square$

OBSERVATION : There is a total non-computable function

$f : \mathbb{N} \to \mathbb{N}$

$$f(n) = \begin{cases} \varphi_n(n) + 1 & \varphi_n(n)\downarrow \quad (n \in W_n) \\ 0 & \varphi_n(n)\uparrow \quad (n \notin W_n) \end{cases}$$

	φ_0	φ_1	φ_2	φ_3 .....
0	$\varphi_0(0)$	$\varphi_1(0)$	$\varphi_2(0)$	..
1	$\varphi_0(1)$	$\varphi_1(1)$	$\varphi_2(1)$	--
2	$\varphi_0(2)$	$\varphi_1(2)$	$\varphi_2(2)$	--
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

- f is total
- f is not computable $\quad f \neq \varphi_n \quad \forall n \in \mathbb{N}$

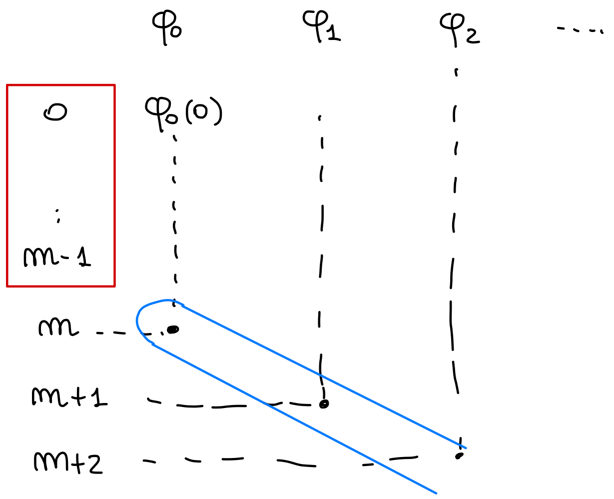
(in fact $\forall n \qquad f(n) \neq \varphi_n(n)$

- $\varphi_n(n)\downarrow \qquad f(n) = \varphi_n(n) + 1 \neq \varphi_n(n)$
- $\varphi_n(n)\uparrow \qquad f(n) = 0 \neq \varphi_n(n)$

EXERCISE : Let $f : \mathbb{N} \to \mathbb{N}$ be any function , $m \in \mathbb{N}$

show that there is a non-computable function $g : \mathbb{N} \to \mathbb{N}$ st.

$$g(n) = f(n) \qquad \forall n < m$$

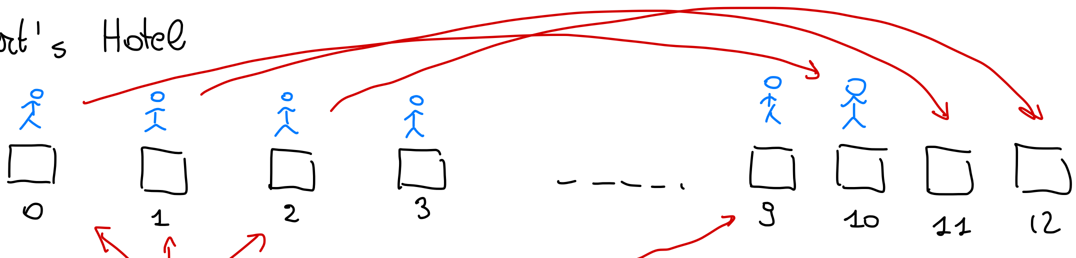


$$g(n) = \begin{cases} f(n) & n < m \\ \varphi_{n-m}(n) + 1 & n \geq m \text{ and } \varphi_{n-m}(n)\downarrow \\ 0 & n \geq m \text{ and } \varphi_{n-m}(n)\uparrow \end{cases}$$

not computable

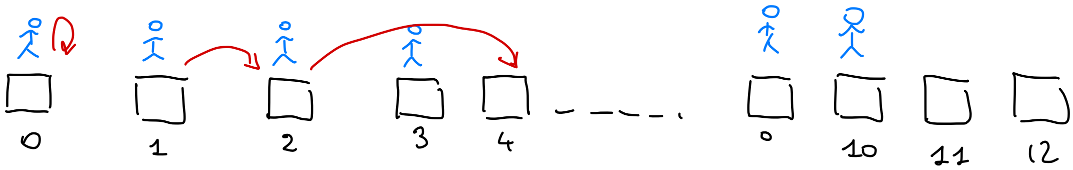
$\forall n.\ \varphi_n \neq g \qquad \varphi_n(n+m) \neq g(n+m)$

Hilbert's Hotel



10 people

if countably many new guests arrive?



$n \to 2n$

Alternative :

$$g(n) = \begin{cases} f(n) & n < m \\ \varphi_n(n)+1 & \text{if } \varphi_n(n)\downarrow,\ n \geq m \\ 0 & \text{if } \varphi_n(n)\uparrow,\ n \geq m \end{cases}$$

g is not computable

$$\varphi_0 \quad \varphi_1 \quad \dots \quad \varphi_{m-1} \quad \overbrace{\varphi_m \quad \varphi_{m+1} \quad \dots}^{g \neq}$$

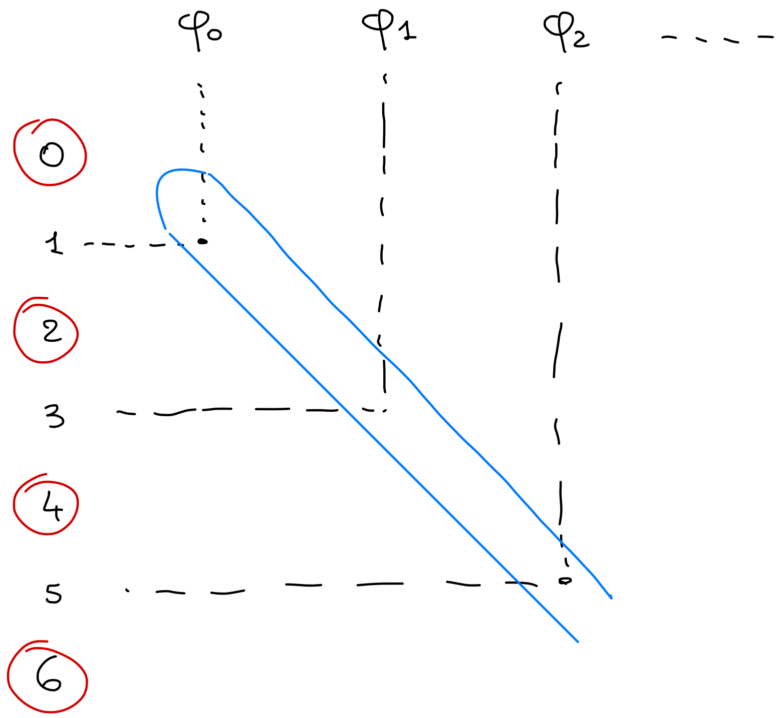
infinitely many repetitions for all computable functions

$g \neq \varphi_n \quad \forall n \geq m$

for all computable functions h $\quad \exists n \geq m \quad h = \varphi_n \neq g$

$\Rightarrow g$ is different from all computable functions $\Rightarrow g$ not computable $\square$

<u>EXERCISE</u>: show that there is a function $g: \mathbb{N} \to \mathbb{N}$ total, not computable s.t. $g(n) = 0 \quad \forall n$ even



$$g(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \varphi_{\frac{n-1}{2}}(n) + 1 & \text{if } n \text{ is odd and } \varphi_{\frac{n-1}{2}}(n)\downarrow \\ 0 & \text{if } n \text{ is odd and } \varphi_{\frac{n-1}{2}}(n)\uparrow \end{cases}$$

→ g is total

→ $g(n) = 0$ for all n even

→ g not computable since $g \neq \varphi_n$ for all $n \in \mathbb{N}$

$$g(2n+1) \neq \varphi_n(2n+1)$$

$$\left(\begin{array}{l} g(1) \neq \varphi_0(1) \\ g(3) \neq \varphi_1(3) \\ \vdots \end{array} \right)$$

<u>EXERCISE</u>: $f_0, f_1, f_2, \dots \quad (f_i)_{i \in \mathbb{N}}$ given

Define $f: \mathbb{N} \to \mathbb{N}$ s.t. $\mathrm{dom}(f) \neq \mathrm{dom}(f_i) \quad \forall i \in \mathbb{N}$

PARAMETRISATION (SMN) THEOREM

Let $f : \mathbb{N}^2 \to \mathbb{N}$ computable

i.e. there $e \in \mathbb{N}$ s.t. $f = \varphi_e^{(2)}$ $\quad (P_e = \gamma^{-1}(e))$

$$f(x,y) = \varphi_e^{(2)}(x,y)$$

Let $x \in \mathbb{N}$ be fixed

$$f_x : \mathbb{N} \to \mathbb{N}$$

$$f_x(y) = f(x,y) = \varphi_e^{(2)}(x,y) \quad \text{is computable}$$

e.g. $f(x,y) = y^x$

$f_0(y) = y^0 = 1$

$f_1(y) = y^1 = y$

$f_2(y) = y^2$

$\vdots$

since for all fixed $x \in \mathbb{N}$ f_x is computable there is $d \in \mathbb{N}$ st.

$$f_x = \varphi_d \text{ --- ——— ——— } = \varphi_{s(e,x)}$$

↑ depends on e, x

hence there is a function $s : \mathbb{N}^2 \to \mathbb{N}$

$$s(e,x) = d$$

The smn theorem says that $s : \mathbb{N}^2 \to \mathbb{N}$ is computable

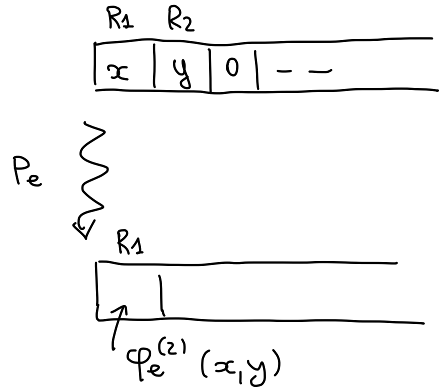
$f(x,y)$ { def P_e (x,y) :
 ⋰ x
 y
 return ...

fix $x = 1$ ⟿

def P_e $(\cancel{x},y)$:
 ⋰ $\cancel{x}$ ← 1
 y
 return ...

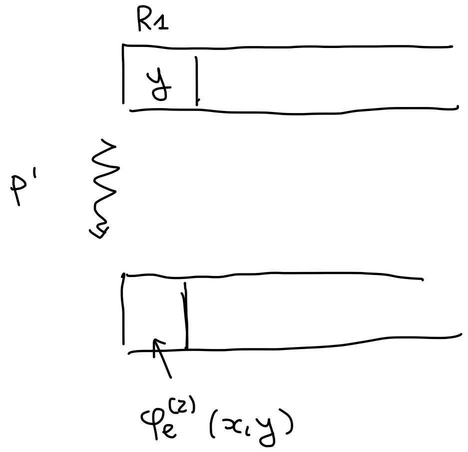
Idea:

given $e \in \mathbb{N}$



for each $x \in \mathbb{N}$ fixed

we want a program P'



what is P' doing?

$$P' \begin{cases} \text{move } y \text{ to } R_2 \\ \text{write } x \text{ to } R_1 \\ \text{execute } P_e = \gamma^{-1}(e) \end{cases}$$

$$S(e,x) = \gamma(P')$$