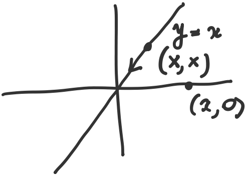
Thursday, 2 November 2023 08:43

Ex 2.7.6

#1 $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}.$$

Prove that limit does not exist.

Sol: Here $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ is defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$.



$$f(x,x) \equiv 0 \xrightarrow[x\to 0]{}$$

$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \xrightarrow[x\to 0]{} 1$$

$$\Rightarrow \nexists \lim_{(x,y)\to\vec{0}} f . \quad ■$$

#2 $\displaystyle\lim_{(x,y)\to\vec{0}} \underbrace{\frac{x^2+y^3}{x^2+y^2}}_{f}.$

Sol: f is defd on $D = \mathbb{R}^2 \setminus \{\vec{0}\}$.

$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \xrightarrow[x\to 0]{} 1$$

$$f(0,y) = \frac{y^3}{y^2} = y \xrightarrow[y\to 0]{} 0$$

$$\Rightarrow \nexists \lim_{(x,y)\to\vec{0}} f. \qquad \blacksquare$$

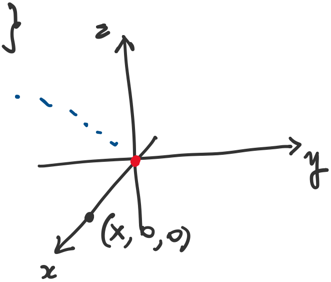
#6 $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \underbrace{\frac{xz}{x^4+y^2+z^2}}_{f(x,y,z)}.$

Sol: f is defd on

$$D = \{(x,y,z)\in\mathbb{R}^3 : x^4+y^2+z^2 \neq 0\}$$

$$\left(\underset{+}{x^4} + \underset{+}{y^2} + \underset{+}{z^2} = 0 \iff \begin{cases} x^4 = 0 \\ y^2 = 0 \\ z^2 = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}\right)$$

$$\Rightarrow \quad D = \mathbb{R}^3 \setminus \{\vec{0}\}$$



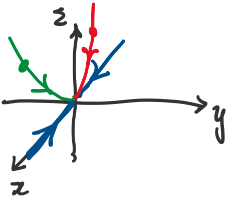
Sections:

$$f(x,0,0) = \frac{x \cdot 0}{x^4 + 0^2 + 0^2} = \frac{0}{x^4} \equiv 0 \to 0$$

$z = x^2$

$y = 0$

$$f(x,0,x^2) = \frac{x \cdot x^2}{x^4 + 0^2 + (x^2)^2}$$



$$= \frac{x^3}{2x^4} = \frac{1}{2x}$$

$$\xrightarrow[x \to 0+]{} +\infty$$

$$\xrightarrow[x \to 0-]{} -\infty$$

$$\Rightarrow \quad \nexists \lim_{(x,y,z) \to \vec{0}} f. \qquad \blacksquare$$

Ex 2.7.7.

#1 Compute

$$\lim_{(x,y)\to\vec{0}} \underbrace{\frac{xy}{\sqrt{x^2+y^2}}}_{f}.$$

Sol: f is defd on $D = \mathbb{R}^2 \setminus \{x^2+y^2=0\}$

$= \mathbb{R}^2 \setminus \{\vec{0}\}$

In pol coords

$x = \rho\cos\vartheta$

$f = \quad y = \rho\sin\vartheta \qquad \dfrac{\rho^{\cancel{2}}\cos\theta\sin\vartheta}{\cancel{\rho}}$

$$= \rho \underbrace{\boxed{\cos\theta\sin\vartheta}}_{-1\le \quad \le 1}$$

$$\Rightarrow \begin{array}{ccccc} -1\cdot\rho & \le & f & \le & 1\cdot\rho \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

$$\begin{array}{c} \|(x,y)\| \to 0 \\ \| \\ \rho \to 0 \\ \Updownarrow \\ (x,y) \to \vec{0} \end{array}$$

$$\Rightarrow \quad \exists \lim_{(x,y)\to\vec{0}} f = 0. \qquad \blacksquare$$

Rmk: What happens if we use pol coords on a non exist. limit ?

2.7.6 #1 $\displaystyle\lim_{(x,y)\to\vec{0}} \frac{x^2-y^2}{x^2+y^2}$.

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$$f = \frac{\cancel{\rho^2}\left(\cos^2\theta - \sin^2\theta\right)}{\cancel{\rho^2}}$$

$$= \cos^2\theta - \sin^2\theta \qquad \rho \to 0$$

$\theta \in [0, 2\pi]$

$\theta = 0 \qquad f = 1$

$\theta = \pi/2 \qquad f = -1$

$\blacksquare$

#2 (modified)

(original text) $\displaystyle\lim_{(x,y)\to\vec{0}} \frac{x^2y^3}{(x^2+y^2)^2}$

(modified text) $\displaystyle\lim_{(x,y)\to\vec{0}} \underbrace{\frac{x^2y^3}{(x^2+2y^2)^2}}_{f}$.

Sol: f is defd on

$$D = \mathbb{R}^2 \setminus \{x^2+2y^2=0\}$$

$$= \mathbb{R}^2 \setminus \{\vec{0}\}.$$

$x = \rho\cos\theta$

$f \quad y = \rho\sin\theta$

$$\frac{\rho^2\cos^2\theta\,\rho^3\sin^3\theta}{(\underbrace{\rho^2+\rho^2}\sin^2\theta)^2}$$

$$x^2+2y^2 = \underbrace{x^2+y^2}_{\rho^2} + \underset{\rho^2\sin^2\theta}{y^2}$$

$$= \frac{\rho^{\cancel{5}}\cos^2\theta\sin^3\theta}{\cancel{\rho^4}(1+\sin^2\theta)^2}$$

$$= \rho\underbrace{\frac{\cos^2\theta\sin^3\theta}{(\underset{\geq 1}{1+\sin^2\theta})^2}}_{F(\theta)}$$

$$f = \rho F(\theta)$$

$$-1 = \frac{-1}{1} \leqslant F(\theta) \leqslant \frac{1}{1} = 1$$

$$\Rightarrow \begin{array}{ccccc} -1\rho & \leqslant & f & \leqslant & 1\cdot\rho \\ \downarrow & & & & \downarrow \\ 0 & & & & 0 \end{array}$$

$$\Rightarrow \quad f \to 0 \qquad \rho \to 0 \Longleftrightarrow (x,y) \to \vec{0}.$$

■

Alt sol:

$$f \Rightarrow \begin{cases} x = \rho\cos\theta \\ \cancel{\sqrt{2}}y = \frac{\rho}{\sqrt{2}}\sin\theta \end{cases} \qquad \frac{\rho^2\cos^2\theta \; \frac{\rho^3}{2^{3/2}}\sin^3\theta}{\rho^4}$$

$$x^2 + 2y^2 = \rho^2 \qquad \left(\rho = \sqrt{x^2+2y^2} \right.$$

$$x^2 + (\sqrt{2}y)^2$$

$$= \frac{\rho^5}{\rho^4} \; \frac{1}{2^{3/2}} \cos^2\theta \; \sin^3\theta$$

$$(x,y) \to \vec{0} \overset{?}{\Longleftrightarrow} \rho \to 0$$

$$\Rightarrow \quad \rho = \underbrace{\sqrt{x^2+2y^2}}_{\substack{\downarrow \\ \geq \\ 0}} \longrightarrow 0$$

$$\Leftarrow \quad \rho = \sqrt{x^2+2y^2} \longrightarrow 0$$

$$\Updownarrow$$

$$x^2+2y^2 \longrightarrow 0 \Leftrightarrow \begin{matrix} x \to 0 \\ y \to 0 \end{matrix}$$

$$\Updownarrow$$

$$(x,y) \downarrow \vec{0}$$

$$\Big(\quad \sqrt{x^2} \leq \sqrt{x^2+2y^2} \longrightarrow 0$$

$$\|$$

$$0 \leq |x| \Rightarrow |x| \longrightarrow 0 \Rightarrow x \to 0$$

$$\underset{\substack{\geq \\ 0}}{\sqrt{2}\,|y|} = \sqrt{2y^2} \leq \underbrace{\sqrt{x^2+2y^2}} = \rho \longrightarrow 0$$

$$\Rightarrow |y| \longrightarrow 0 \Rightarrow y \to 0$$

Exercise Generalize this arg showing that

- $(x,y) \to (0,0) \Leftrightarrow \sqrt{ax^2+by^2} \longrightarrow 0$

  $a, b > 0$

- $(x_1, \text{—}, x_d) \longrightarrow \vec{0}$

$$\Leftrightarrow \quad \sqrt{a_1 x_1^2 + a_2 x_2^2 + \dots + a_d x_d^2} \longrightarrow 0$$

$$\forall\, a_1, \dots, a_d .$$

- Show that

$$\|(x_1, \dots, x_d)\|_* := \sqrt{\sum_{j=1}^{d} a_j x_j^2}$$

where $a_1, a_2, \dots, a_d$ are pos (>0) constants is a norm on $\mathbb{R}^d$, that is it fulfils positivity, vanishing, homog, triang ineq. $\blacksquare$

So since

$$f = \rho \cdot \boxed{\frac{1}{2^{3/2}} \cos^2\theta \, \sin^3\theta}$$

$$-1/2^{3/2} \le \qquad\qquad \le 1/2^{3/2}$$

$$-\frac{1}{2^{3/2}}\rho \le f \le \frac{1}{2^{3/2}}\rho$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$0 \qquad\qquad 0$

$\Rightarrow \; f \to 0$ when $\rho \to 0 \iff (x,y) \to \vec{0}$ ▨

#4 $$\lim_{(x,y)\to\vec{0}} \underbrace{\frac{x\sqrt{|y|}}{\sqrt[3]{x^4+y^4}}}_{f} .$$

Sol: f is clearly defd on

$$D = \mathbb{R}^2 \setminus \{x^4+y^4=0\}$$
$$= \mathbb{R}^2 \setminus \{\vec{0}\}.$$

Using pol coords

$x = \rho\cos\theta$
$y = \rho\sin\theta$

$$f = \frac{\boxed{\rho}\cos\theta\,\overset{\rho^{1/2}\sqrt{|\sin\theta|}}{\sqrt{|\rho\sin\theta|}}}{\sqrt[3]{\rho^4\cos^4\theta+\rho^4\sin^4\theta}}$$

$$= \frac{\boxed{\rho^{3/2}}}{\boxed{\rho^{4/3}}}\boxed{\frac{\cos\theta\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta+\sin^4\theta}}}$$

$$= \rho^{\frac{3}{2}-\frac{4}{3}=\frac{1}{6}} \qquad F(\theta)$$

$$\boxed{f = \underset{\substack{\downarrow\\ 0}}{\rho}^{\,1/6}\; F(\theta)}$$

$$(x,y) \to \vec{0} \iff \rho \to 0 \qquad \theta \in [0, 2\pi]$$

$$F(\theta) = \frac{-1 \leq \boxed{\cos\theta\sqrt{|\sin\theta|}} \leq 1}{\sqrt[3]{\cos^4\theta + \sin^4\theta}}$$

$$-\frac{1}{\sqrt[3]{\cos^4\theta + \sin^4\theta}} \leq F(\theta) \leq \underbrace{\frac{1}{\sqrt[3]{\cos^4\theta + \sin^4\theta}}}_{g(\theta)}$$

Pb: $g(\theta) > 0$ $\forall \theta \in [0, 2\pi]$ ($g(\theta) = 0$ is imposs.)

What about $\min\limits_{\theta \in [0,2\pi]} g(\theta)$?

Weierstrass thm says that

$$\exists \min_{\theta\in[0,2\pi]} g(\theta) = \underbrace{g(\theta^*)}_{\overset{\shortparallel}{C}} > 0$$

$$\left(g \in \mathcal{C}([0,2\pi]) \right)$$

$$\Rightarrow \sqrt[3]{\cos^4\theta + \sin^4\theta} \geq C \qquad \forall \theta \in [0,2\pi]$$

$$\Rightarrow \quad F \leq \frac{1}{g(\theta)} \leq \frac{1}{C} \qquad //$$

$$F \geq -\frac{1}{g(\theta)} \geq -1/C$$

$$\begin{tikzcd}
\Rightarrow -\frac{1}{C}\rho^{1/6} \arrow[d] & \leq f \leq & \frac{1}{C}\rho^{1/6} \arrow[d] & \\
0 & & 0 & \rho \to 0
\end{tikzcd}$$

$$\Rightarrow \quad f \to 0 \qquad \rho \to 0 \qquad \blacksquare$$

Let's draw some conclusion:

To show that

$$\lim_{(x,y)\to\vec{0}} f(x,y) = 0$$

we need a bound

$$-\varphi(\rho) \leqslant f \leqslant \varphi(\rho)$$

where $\varphi(\rho) \to 0$ when $\rho \to 0$.

$(\rho = \|(x,y)\|)$.

What if

$$\lim_{(x,y)\to\vec{0}} f = \ell \in \mathbb{R} \ ?$$

$$\Leftrightarrow \lim_{(x,y)\to\vec{0}} (f-\ell) = 0$$

so we need a bound

$$-\varphi(\rho) \leqslant f - \ell \leqslant \varphi(\rho)$$

with $\varphi(\rho) \to 0$ $\rho \to 0$.

What if

$$\lim_{(x,y)\to(x_0,y_0)} f = \ell$$

$$\Updownarrow$$

$$(x,y)-(x_0,y_0)\to\vec{0}$$

$$(\underbrace{x-x_0}_{u},\ \underbrace{y-y_0}_{v})\to\vec{0}$$

$$\Updownarrow$$

$$\lim_{(u,v)\to\vec{0}} \overbrace{\Big(f(u+x_0,\ v+y_0)-\ell\Big)}^{g(u,v)} = 0$$

Do 2.7.7 #5

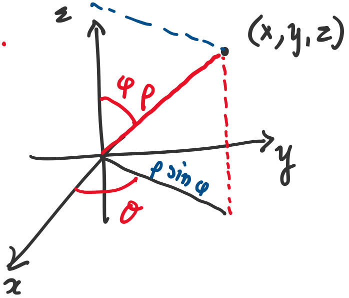
2.7.8

#2 $$\lim_{(x,y,z)\to\vec{0}} \underbrace{\frac{xyz}{x^2+y^2+z^2}}_{f}.$$

Sol: f is defd on $D = \mathbb{R}^3\setminus\{\vec{0}\}$.

To compute the lim we use

spherical coords.



$$\begin{cases} z = \rho\cos\varphi \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases} \qquad \begin{aligned} &\rho \geq 0 \\ &\varphi \in [0,\pi] \\ &\theta \in [0,2\pi] \end{aligned}$$

$$f \overset{\text{sph coords}}{=} \frac{\rho^{\cancel{3}}\,\sin^2\varphi\cos\theta\sin\theta\cos\varphi}{\cancel{\rho^2}}$$

$$f = \boxed{\rho}\ \boxed{\sin^2\varphi\cos\theta\sin\theta\cos\varphi}$$

$$-1 \leq \qquad\qquad\qquad\qquad \leq 1$$

$$\Rightarrow \qquad -\underset{\substack{\downarrow \\ 0}}{\rho} \leq f \leq \underset{\substack{\downarrow \\ 0}}{\rho}$$

$$(x,y,z) \longrightarrow \vec{0} \quad \Leftrightarrow \quad \rho \to 0 \quad \varphi \in [0,\pi] \quad \theta \in [0,2\pi]$$

$\Rightarrow \quad f \to 0$ when $(x,y,z) \longrightarrow \vec{0}$. ∎