Monday, 30 October 2023 08:37

Ex. 2.2.6. Discuss existence and value (if any) of

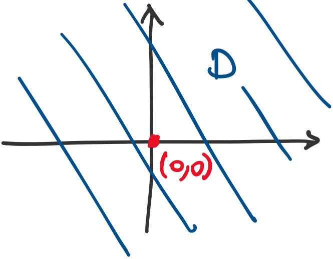
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

Sol: Here $f(x,y) = \frac{xy^2}{x^2+y^4}$, defined on

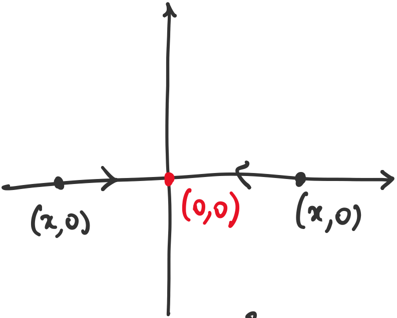
$$D = \{(x,y)\in\mathbb{R}^2 : x^2+y^4 \neq 0\}$$

$$\underset{+}{x^2} + \underset{+}{y^4} = 0 \iff \begin{cases} x^2 = 0 \\ y^4 = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

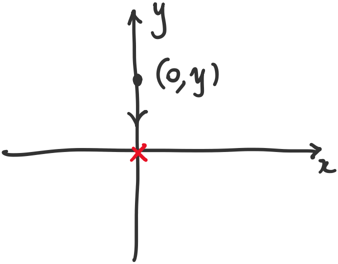
$$= \mathbb{R}^2 \setminus \{(0,0)\}.$$



Let's compute some section of f

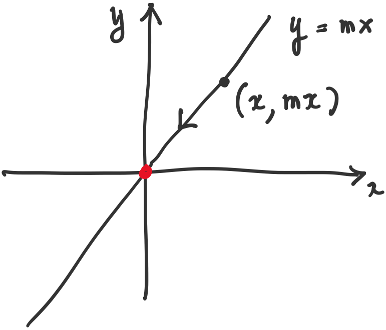


$$\underset{\substack{\uparrow \\ x\text{ - sect. of } f}}{f(x,0)} = \frac{x \cdot 0^2}{x^2 + 0^4} = \frac{0}{x^2} = 0 \xrightarrow[\substack{(x,0) \\ \downarrow \\ \vec{0}}]{} 0$$



$$f(0,y) = \frac{0 \cdot y^2}{0^2 + y^4} = \frac{0}{y^4} \equiv 0 \xrightarrow[(0,y) \to (0,0)]{} 0$$

y - set



$$f(x,mx) = \frac{x \cdot (mx)^2}{x^2 + (mx)^4} = \frac{m^2 x^3}{x^2 + m^4 x^4}$$

$$= \frac{m^2 x^{\cancel{3}}}{\cancel{x^2}(1 + m^4 x^2)}$$

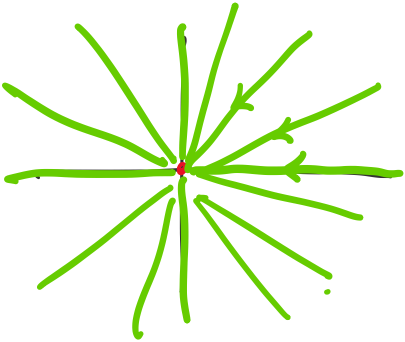
$$= \frac{m^2 x}{1 + m^4 x^2} \xrightarrow[(x,mx) \to (0,0)]{} \frac{m^2 \cdot 0}{1 + m^4 \cdot 0}$$

$$\Updownarrow$$

$$x \to 0$$

$$= \frac{0}{1} = 0$$

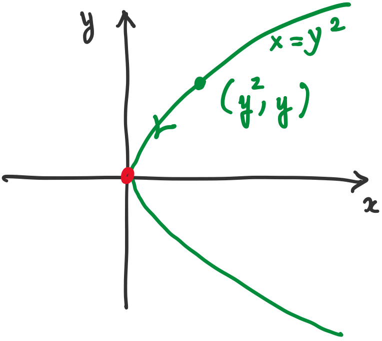
$$\Rightarrow \quad f(x, mx) = \frac{m^2 x}{1 + m^4 x^2} \xrightarrow[(x,mx)\to(0,0)]{} 0$$



$$f \longrightarrow 0$$

$$\Rightarrow \quad \exists \lim_{(x,y)\to(0,0)} f = 0 \quad ?$$

$x = y^2$



$$f(x,y) = \frac{xy^2}{x^2 + y^4}$$

$$\underbrace{(y^2, y) \to (0,0)} \quad \Leftrightarrow \quad y \to 0$$

$$f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{1\,\cancel{y^4}}{2\cancel{y^4}} \equiv \begin{array}{c} 1/2 \\ \Big\downarrow {\scriptstyle y \to 0} \\ 1/2 \end{array}$$

Conclusion: $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$. $\blacksquare$

Ex. 2.2.8

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}.$$

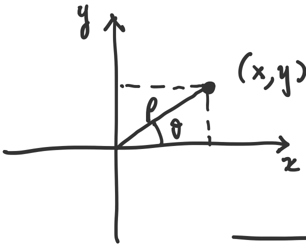
Sol: $f(x,y) = \dfrac{xy^2}{x^2+y^2}$, $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

Sections:

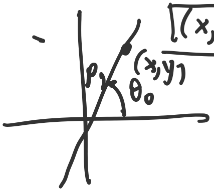
$$f(x,0) = \frac{0}{x^2} \equiv 0 \xrightarrow[(x,0)\to\vec{0}]{} \boxed{0}\checkmark$$

(unique information drawn: if lim exists it must be 0)

$$\left\ulcorner \lim_{x\to 0} \frac{\boxed{x^2} + x^3}{\boxed{3x^2} - x^4} \right\lrcorner$$



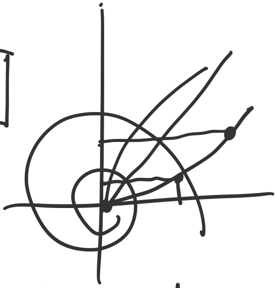
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$



$$\boxed{(x,y)} \rightsquigarrow \boxed{(\rho, \boxed{\theta})}$$

$$\downarrow$$

$$0$$



Let's rewrite f in pol coords

$$f(x,y) \overset{\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}}{=} \frac{\rho\cos\theta(\rho\sin\theta)^2}{\rho^2}$$

$$x^2 + y^2 = \rho^2$$

$$\left(\rho = \|(x,y)\| = \sqrt{x^2+y^2}\right)$$

$$= \frac{\boxed{\rho^{\cancel{3}} \cos\theta \sin^2\theta}}{\cancel{\rho^2}} \qquad N$$

$$(x,y) \to (0,0) \iff \rho \to 0, \qquad \theta \in [0, 2\pi]$$

$$\boxed{f = \rho \underbrace{\cos\theta \sin^2\theta}_{-1 \le \qquad \le 1}}$$

$$\boxed{-\underset{\substack{\downarrow \\ 0}}{\rho} \le f \le \underset{\substack{\downarrow \\ 0}}{\rho}}$$

$$(x,y) \to (0,0) \Updownarrow \rho \to 0$$

$\Rightarrow$ (policemen thm)

$$\lim_{(x,y)\to(0,0)} f = 0$$

Ex 2.7.7.

#3 $$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}.$$

Sol: Here $f(x,y) = \dfrac{x^3 - y^3}{x^2+y^2}$ is defd on $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

Sects :

$$f(x,0) = \frac{x^3}{x^2} = x \xrightarrow[\substack{(x,0)\to(0,0)\\ \Updownarrow \\ x\to 0}]{} 0$$

$$f(0,y) = \frac{-y^3}{y^2} = -y \xrightarrow[\substack{(0,y)\to(0,0)\\ \Updownarrow \\ y\to 0}]{} 0$$

$$f(x,x) = \frac{0}{2x^2} \equiv 0 \xrightarrow[(x,x)\to(0,0)]{} 0 .$$

Let's write f in pol coords

$$f(x,y) \overset{\substack{x=\rho\cos\theta\\ y=\rho\sin\theta}}{=} \frac{\rho^{\cancel{3}}(\cos^3\theta - \cancel{\rho^3}\sin^3\theta)}{\cancel{\rho^2}}$$

$$\boxed{f = \rho\left(\underset{-2\le \qquad\qquad\qquad \le 2}{\cos^3\theta - \sin^3\theta}\right)}$$

$$(x,y)\to(0,0) \iff \rho\to 0,\quad \theta\in[0,2\pi]$$

$$\Downarrow$$

$$\begin{array}{ccccc} -2\rho & \le & f & \le & 2\rho \\ \downarrow & & & & \downarrow \\ 0 & & & & 0 \end{array} \qquad \rho\to 0$$

$$\Rightarrow \quad f \longrightarrow 0. \qquad \blacksquare$$

Do 2.7.7

# 1, 2

# 4, 5 (not easy).