COMPUTABILITY (30/10/2023)

* Class of partial recursive functions $\mathcal{R}$

least rich class of functions i.e. least class of functions

- → including the BASIC FUNCTIONS
- → closed under
  1. COMPOSITION
  2. PRIMITIVE RECURSION
  3. UNBOUNDED MINIMALISATION

Theorem : $\mathcal{R} = \mathcal{C}$

proof

$(\mathcal{R} \subseteq \mathcal{C})$ $\quad$ $\mathcal{C}$ is rich

$(\mathcal{C} \subseteq \mathcal{R})$

Let $f : \mathbb{N}^k \to \mathbb{N}$ be a function in $\mathcal{C}$

and let P a URM - program for f

Define

$$\begin{cases} c_P^1 : \mathbb{N}^{k+1} \to \mathbb{N} \\ c_P^1(\vec{x}, t) = \text{content of register } R_1 \text{ after } t \text{ steps of } P(\vec{x}) \end{cases}$$

$$\begin{cases} J_P : \mathbb{N}^{k+1} \to \mathbb{N} \\ J_P(\vec{x}, t) = \begin{cases} \text{instruction to be executed after } t \text{ steps of } P(\vec{x}) \\ 0 \qquad \text{if } P(\vec{x}) \text{ halts in } t \text{ or fewer steps} \end{cases} \end{cases}$$

Then

$$f(\vec{x}) = c_P^1(\vec{x}, \mu t.\ J_P(\vec{x}, t))$$

We conclude by proving $c_P^1, J_P \in \mathcal{R}$

program P (std form) for f

$\rightarrow I_1$

$\rightarrow I_2$

$\vdots$

I_s

memory

r_1	r_2	r_3	----	r_m	0	...	0

$$C = \prod_{i \geq 1} p_i^{r_i} = \prod_{i=1}^{m} p_i^{r_i}$$

$$r_i = (C)_i$$

1	2	3				
2	0	1	0	0	0	---

$$C = p_1^2 \cdot p_2^0 \cdot p_3^1 \cdot \underset{1}{\overset{\shortparallel}{p_4^0}} \cdot \underset{1}{\overset{\shortparallel}{p_5^0}}$$

$$= 2^2 \, 3^0 \, 5^1 = 20$$

$$\begin{cases} C_P : \mathbb{N}^{k+1} \to \mathbb{N} \\ C_P(\vec{x}, t) = \text{content of memory after } t \text{ steps of } P(\vec{x}) \end{cases}$$

$$\begin{cases} J_P : \mathbb{N}^{k+1} \to \mathbb{N} \\ J_P(\vec{x}, t) = \begin{cases} \text{instruction to be executed after } t \text{ steps of } P(\vec{x}) \\ 0 \quad \text{if } P(\vec{x}) \text{ halts in } t \text{ or fewer steps} \end{cases} \end{cases}$$

we define J_P, C_P by primitive recursion

$$\begin{cases} C_P(\vec{x}, 0) = \prod_{i=1}^{k} p_i^{x_i} \\ J_P(\vec{x}, 0) = 1 \end{cases}$$

x_1	--	x_k	0	---	0	

recursion cases

we define $C_P(\vec{x}, t+1)$

$J_P(\vec{x}, t+1)$

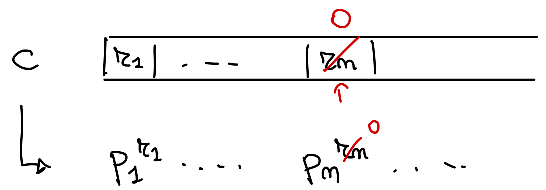
using

$C_P(\vec{x}, t) = C$

$J_P(\vec{x}, t) = J$

NOTATION

$$C_P(\vec{x}, t+1) = \begin{cases} qt\left(p_m^{(C)_m}, C\right) & \text{if } 1 \leq J \leq \ell(P) \text{ and } I_J = Z(m) \\ p_m \cdot C & \text{if } 1 \leq J \leq \ell(P) \text{ and } I_J = S(m) \\ p_n^{(C)_m} \cdot qt\left(p_m^{(C)_m}, C\right) & \text{if } 1 \leq J \leq \ell(P) \text{ and } I_J = T(m,n) \\ C & \text{otherwise } (J = 0 \text{ or } 1 \leq J \leq \ell(P) \text{ and } I_J = J(m,n,u)) \end{cases}$$



$r_m = (C)_m$

$$J_P(\vec{x}, t) = \begin{cases} J+1 & \text{if } 1 \leq J < \ell(P) \text{ and } I_J = S(m), Z(m), T(m,n) \text{ or } \left(\begin{array}{l} I_J = J(m,n,u) \\ (C)_m \neq (C)_n \end{array} \right) \\ u & \text{if } 1 \leq J \leq \ell(P) \text{ and } I_J = J(m,n,u) \text{ and } (C)_m = (C)_n \text{ and } u \leq \ell(P) \\ 0 & \text{otherwise} \end{cases}$$

Hence $J_P, C_P \in \mathcal{R}$

and thus

$$f(\vec{x}) = \left(C_P\left(\vec{x}, \mu t. J_P(\vec{x}, t)\right) \right)_1$$

therefore $f \in \mathcal{R}$

$\square$

* Primitive Recursive Functions

PR = least class of functions which

→ includes the basic functions

→ closed under

① composition

② primitive recursion ← for loop

③ minimisation ← while loop

$$PR \underset{?}{\subsetneq} R \cap Tot$$

$$\| \qquad\qquad \|$$

$$\mathcal{C}_{for} \qquad\qquad \mathcal{C}_{for,\,while}$$

$$\not\ni \qquad\qquad \ni$$

$$\psi \qquad\qquad \psi$$

Ackermann's Function

$\psi : \mathbb{N}^2 \to \mathbb{N}$

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x, \underbrace{\psi(x+1,y)}_{u}) \end{cases}$$

$(x+1,0) >_{lex} (x,1)$

$(x+1, y+1) >_{lex} (x+1, y)$

$(x+1, y+1) >_{lex} (x, u)$

$(\mathbb{N}^2, \leq_{lex})$ $\quad (x,y) \leq_{lex} (x',y')$ if $x < x'$ or $(x = x')$ and $(y \leq y')$

$(1000, 1000\,000) <_{lex} (1001, 0)$

$(1000, 1000\,000) >_{lex} (1000, 0)$

$f : \mathbb{Z} \to \mathbb{Z}$

$$f(z) = \begin{cases} 0 & z \geq 0 \\ f(z-1) & z < 0 \end{cases}$$

$f(-1) = f(-2) = f(-3) = \ldots$

* <u>partially ordered set</u> (poset)

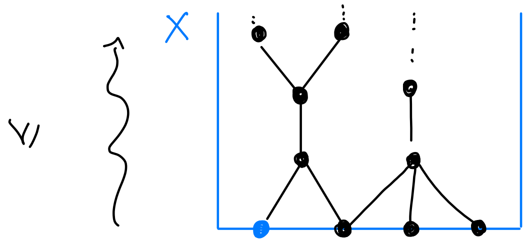
$(D, \leq)$ $\quad \leq$ reflexive $\quad x \leq x$

antisymmetric $\quad x \leq y$ and $y \leq x \Rightarrow x = y$

transitive $\quad x \leq y$ and $y \leq z \Rightarrow x \leq z$

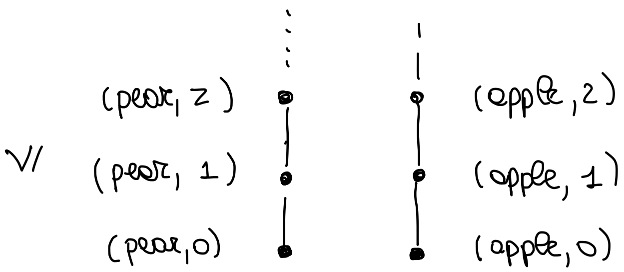
* <u>well founded posets</u>

$(D, \leq)$ is well-founded if $\forall X \subseteq D \quad X \neq \emptyset$ has a minimal element



$d \in X$
minimal s.t. if $d' \leq d$ then $d' = d$

$$D = \{ (pear, n), (apple, n) \mid n \in \mathbb{N} \}$$



minimal

$(x, y) \leq (x', y')$
if $(x = x')$
and $(y \leq y')$

$\mathbb{Z}$ well-founded? NO

$\mathbb{N}$ " " ? yes

<u>NOTE</u>: $(D, \leq)$ well-founded if and only if there is no infinite descending chain in D
$d_0 > d_1 > d_2 > \ldots$

* $(\mathbb{N}^2, \leq_{lex})$ is well founded

let $X \subseteq \mathbb{N}^2 \quad X \neq \emptyset$

$x_0 = \min \{ x \mid \exists y. (x, y) \in X \}$

$y_0 = \min \{ y \mid (x_0, y) \in X \}$

$\rightsquigarrow \quad (x_0, y_0) = \min X$

* Induction $\quad P(n) \quad n \in \mathbb{N}$

$$\underbrace{P(0) \text{ and assuming } P(n) \text{ you can deduce } P(n+1)}$$

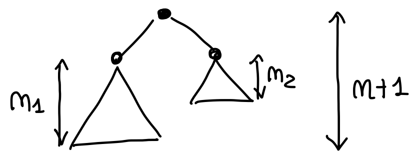
$$\Downarrow$$

$$P(n) \text{ holds for all } n$$

- A binary tree with heigth n has at most $2^{n+1}-1$ nodes

$(n=0)$ • number of nodes $= 1 \leq 2^{0+1}-1 = 2-1 = 1$

$(n \to n+1)$



$n_1, n_2 < n+1$

the inductive hyp. is only on n ....

you "can't" conclude

- Complete induction

to prove that $P(n)$ holds for all $n \in \mathbb{N}$

$\Uparrow$

show $\boxed{\text{for all } n \text{, assuming } P(n') \text{ for all } n' < n \text{ then } P(n)}$

- Well-founded induction

$(D, \leq)$ well-founded order

$P(x)$ property over D

$$\underbrace{\text{if for all } d \in D \text{, assuming } \forall d' < d \;\; P(d') \text{ I can can conclude } P(d)}$$

$$\Downarrow$$

$$\forall d \in D \;\; P(d)$$

$\boxed{① \ \psi \text{ is total}}$

$\forall (x,y) \in \mathbb{N}^2 \qquad \psi(x,y)\downarrow$

proceed by well-founded induction on $(\mathbb{N}^2, \leq_{lex})$

proof

let $(x,y) \in \mathbb{N}^2$, assume $\forall (x',y') <_{lex} (x,y) \quad \psi(x',y')\downarrow$

we want to show $\psi(x,y)\downarrow$

$$\begin{cases} \psi(0,y) = y+1 \\ \psi(x+1,0) = \psi(x,1) \\ \psi(x+1,y+1) = \psi(x, \underbrace{\psi(x+1,y)}_{u}) \end{cases}$$

3 cases

$(x=0) \qquad \psi(x,y) = \psi(0,y) = y+1 \downarrow$

$(x>0, y=0) \qquad \psi(x,0) = \psi(\underbrace{x-1,1}_{})\downarrow$

$(x-1,1) <_{lex} (x,y)$ hence $\psi(x-1,1)\downarrow$ by ind. hyp.

$(x>0, y>0) \qquad \psi(x,y) = \psi(x-1, \psi(\underbrace{x,y-1}_{<_{lex}(x,y)})) = \psi(\underbrace{x-1,u}_{<_{lex}(x,y)})\downarrow$ by ind. hyp.

$<_{lex}(x,y) \rightsquigarrow \psi(x,y-1)\downarrow = u$

ind. hyp.

$\square$

$(\mathbb{N}^2, \leq_{lex})$

$\underbrace{(0,0) \ (0,1) \ (0,2) \ \ldots}_{\mathbb{N}} \quad \underbrace{(1,0) \ (1,1) \ (1,2) \ \ldots}_{\mathbb{N}} \quad \underbrace{(2,0) \ (2,1) \ (2,2) \ \ldots}_{\mathbb{N}}$

$\boxed{② \ \psi \in \mathcal{R} = \mathcal{C}}$

$$\psi(1,1) = \psi(0, \underbrace{\psi(1,0)}_{\psi(0,1) = 2}) = \psi(0,2) = 3$$

$(1,1,3) \quad (0,2,3) \quad (1,0,2) \ , \ (0,1,2)$

valid set of triples : informally

$(x,y,z) \in \mathbb{N}^3$ $\rightarrow$ $z = \varphi(x,y)$

$\rightarrow$ S contains all triples needed to compute $\varphi(x,y)$

$$\begin{cases} \varphi(0,y) = y+1 \\ \varphi(x+1,0) = \varphi(x,1) \\ \varphi(x+1,y+1) = \varphi(x, \underbrace{\varphi(x+1,y)}_{u}) \end{cases}$$

formally $S \subseteq \mathbb{N}^3$ valid if

1. $(0,y,z) \in S \Rightarrow z = y+1$
2. $(x+1,0,z) \in S \Rightarrow (x,1,z) \in S$
3. $(x+1,y+1,z) \in S \Rightarrow \exists u.\ (x+1,y,u) \in S \wedge (x,u,z) \in S$

you can prove that $\forall (x,y,z) \in \mathbb{N}^3$

$\varphi(x,y) = z$ iff $\exists S \subseteq \mathbb{N}^3$ a valid finite set of triples s.t. $(x,y,z) \in S$

then

$$\varphi(x,y) = \text{“}\ \mu \underbrace{(S,z)}\ .\ \left(S \subseteq \mathbb{N}^3 \text{ valid finite set of triples } \wedge (x,y,z) \in S \right) \text{”}$$

encode as a number

$S = \{(x_1,y_1,z_1), (x_2,y_2,z_2), \dots, (x_m,y_m,z_m)\}$

$$\{\underbrace{\pi(\pi(x_1,y_2),z_1)}_{k_1}, \ \dots\ \underbrace{\pi(\pi(x_m,y_m),z_m)}_{k_m}\}$$

$$\prod_{i=1}^{m} p_i^{k_i}$$

$\leadsto$ $\varphi \in \mathcal{R} = \mathcal{C}$

③ $\psi \notin PR$

successor

$x + y$ $\quad$ $x + 0 = x$

$\quad\quad\quad$ $x + (y+1) = (x+y) + 1$

$x * y$ $\quad$ $x * 0 = 0$

$\quad\quad\quad$ $x * (y+1) = (x * y) + x$

x^y $\quad$ $x^0 = 1$

$\quad\quad\quad$ $x^{y+1} = (x^y) * x$

$\vdots$

nesting primitive recursion

idea: ψ brings the above to the limit

$$\begin{cases} \psi(0, y) = y + 1 \\ \psi(x+1, 0) = \psi(x, 1) \\ \psi(x+1, y+1) = \psi(x, \psi(x+1, y)) \end{cases}$$

consider x as a "fixed" parameter

$$\psi(x, y) = \psi_x(y)$$

$$\begin{aligned} \psi_{x+1}(y) &= \psi_x(\psi_{x+1}(y-1)) \\ &= \psi_x(\psi_x(\psi_{x+1}(y-2))) \\ &\vdots \\ &= \underbrace{\psi_x \psi_x \cdots \psi_x}_{y} \underbrace{\psi_{x+1}(0)}_{\psi_x(1)} \\ &= \psi_x^{y+1}(1) \end{aligned}$$

roughly: increasing x to $x+1$ requires iterating the function ψ_x

$\rightarrow$ increases the number of nested primitive recursion

→ the full function would require infinitely many nested primitive recursions

Some more ideas....

concretely:

$\varphi_0(y) = y+1$

$\varphi_1(y) = \varphi_0^{y+1}(1) = y+2$

$\varphi_2(y) = \varphi_1^{y+1}(1) = 2(y+1)+1 = 2y+3 \approx 2y$

$\varphi_3(y) = \varphi_2^{y+1}(1) \cong 2^y$

$\varphi_4(y) = \varphi_3^{y+1}(1) \cong 2^{2^{2^{2^{\cdot^{\cdot}}}}}\Big\} y$

es: $\varphi_0(1) = 2$

$\varphi_2(1) = 5$

$\varphi_3(1) = 13$

$\varphi_4(1) \cong 2^{16}$

$\varphi_4(2) \cong 2^{2^{16}} \cong 10^{6400}$

ONE CAN PROVE: Given a function $f: \mathbb{N}^m \to \mathbb{N} \in PR$ and a program P computing f using only "for-loops" (primitive recursion) if J is the maximum level of nesting of for-loops

$$f(\vec{x}) < \varphi_{J+1}(\max\{x_i\})$$

Now, assume $\varphi \in PR$, let J be the level of nesting of for-loops (of primitive recursive defs) for computing φ

$\forall (x,y)$

$$\varphi(x,y) < \varphi_{J+1}(\max\{x,y\})$$

let $x = y = J+1$

$$\varphi(J+1, J+1) < \varphi_{J+1}(J+1) = \varphi(J+1, J+1)$$

contradiction

$\Rightarrow \quad \varphi \notin PR$