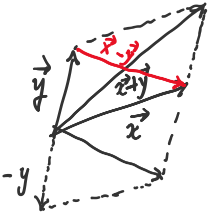
Thursday, 26 October 2023 08:38

Ex 2.7.1

Prove the parallelogram id

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$$



Sol: Recall that

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\,\vec{x}\cdot\vec{y}$$

$+$

$$\|\vec{x}-\vec{y}\|^2 = \|\vec{x}+(-\vec{y})\|^2$$

$$= \|\vec{x}\|^2 + \overset{\overset{\|\vec{y}\|^2}{\|}}{\overset{|-1|^2\|\vec{y}\|^2}{\overset{\|}{\|-\vec{y}\|^2}}} + 2\,\overset{\sum_j x_j(-y_j)}{\overset{\|}{\vec{x}\cdot(-\vec{y})}}$$

$$= \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\,\vec{x}\cdot\vec{y}$$

Summing the identities

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 + 0.$$

■

Ex 2.7.2.

i) $\vec{x}_n = (e^{-n}, 1)$

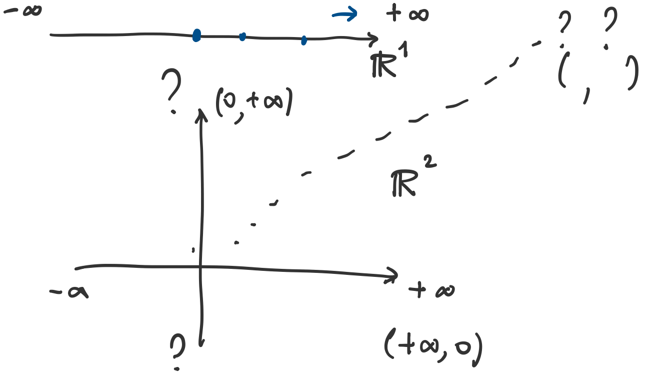
$$\left.\begin{array}{l} x_{1,n} = e^{-n} \longrightarrow 0 \\ x_{2,n} = 1 \longrightarrow 1 \end{array}\right\} \Rightarrow \vec{x}_n \longrightarrow (0,1)$$

ii) $$\vec{x}_n = \Big(\underset{\substack{\downarrow\\0}}{\frac{1}{n}}, \underset{\substack{\downarrow\\0}}{\frac{1}{n^2}}, \underset{\substack{\downarrow\\0}}{\sin \underset{\substack{\downarrow\\0}}{\frac{1}{n}}}\Big) \longrightarrow (0,0,0).$$

We know that a numerical seq $(x_n) \subset \mathbb{R}$ can

- $\boxed{\text{have a finite limit } l \in \mathbb{R}}$
- $\boxed{\text{have an infinite limit } \quad x_n \to +\infty \text{ or } x_n \to -\infty}$
- be not convergent (ex $x_n = (-1)^n$)

We want now to define the case of a seq going to "infinity"



Def: We say that

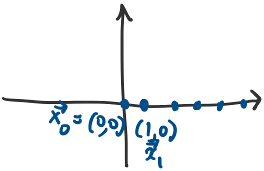
$$\lim_{n\to\infty} \vec{x}_n = \infty_d$$

$((\vec{x}_n) \subset \mathbb{R}^d)$ if

$$\|\vec{x}_n\| \longrightarrow +\infty .$$

Examples:

- $\vec{x}_n = (n,0)$

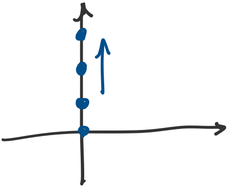


$$\|\vec{x}_n\| = \sqrt{n^2 + 0^2}$$

$$= \sqrt{n^2} \underset{n \in \mathbb{N}}{=} n \xrightarrow[n \to +\infty]{} +\infty$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow \infty_2$$

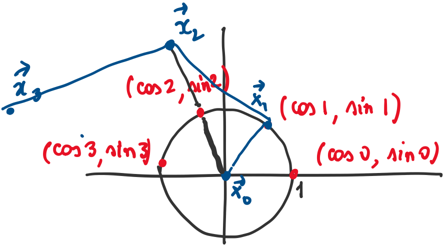
- $\vec{x}_n = (0,n)$



$\|\vec{x}_n\| = n \longrightarrow +\infty$

$\vec{x}_n \longrightarrow \infty_2$.

- $\vec{x}_n = (n\cos n, n\sin n) = n(\cos n, \sin n)$



$$\|\vec{x}_n\| = \sqrt{n^2\cos^2 n + n^2\sin^2 n} = \sqrt{n^2(\cos^2 n + \sin^2 n)}$$

$$= \sqrt{n^2} = n \longrightarrow +\infty$$

$n \in \mathbb{N}$

$$\Rightarrow \quad \vec{x}_n \longrightarrow \infty_2$$

But notice what happens to the comp.

$$x_{1,n} = n\cos n$$

(the behaviour is similar to $n \cdot (-1)^n$

$$\not\downarrow$$

$$x_{2,n} = n \sin n \not\to$$

2.7.2

v) $\vec{x}_n = \left(\operatorname{th} n, \ \frac{\log n}{n}, \ \frac{\sin n}{n} \right)$

$$x_{1,n} = \operatorname{th} n = \frac{\operatorname{Sh} n}{\operatorname{Ch} n} = \frac{\frac{e^n - e^{-n}}{2}}{\frac{e^n + e^{-n}}{2}} =$$

$$= \frac{e^n - e^{-n}}{e^n + e^{-n}} = \frac{\cancel{e^n}\left(1 - e^{-2n}\right)}{\cancel{e^n}\left(1 + e^{-2n}\right)}$$

$$\longrightarrow 1$$

$$x_{2,n} = \frac{\log n}{n} \longrightarrow 0 \qquad \log n = o(n)$$

$$x_{3,n} = \frac{\sin n}{n} = \underset{\underset{0}{\swarrow}}{\frac{1}{n}}\ \underbrace{\sin n}_{\text{bded}} \longrightarrow 0$$

$$| \quad | \leq 1$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow (1,0,0). \qquad \blacksquare$$

iv)

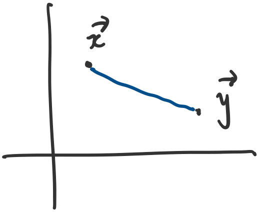
$$\vec{x}_n = \begin{array}{ccccc} (& 1, & \overset{1_n}{\boxed{1 + 1/n}}, & n &) \\ & \downarrow & \downarrow & \downarrow & \\ & 1 & 1 & \boxed{+\infty} & \end{array}$$

$$\|\vec{x}_n\| = \sqrt{\underbrace{1^2 + 1_n^2 + n^2}_{\to +\infty}} \longrightarrow +\infty$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow \infty_3. \qquad \blacksquare$$

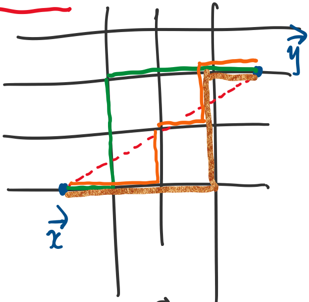
Do ex 2.7.3. (and, of course, what remains of 2.7.2).

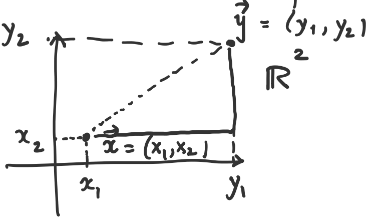
A final rmk on norms. We introduced Eucl. norm to measure dist between vectors



This is not the unique way we can introduce a measure of dist.

Manhattan norm





$$\mathrm{dist}_M(\vec{x},\vec{y}) := |x_1 - y_1| + |x_2 - y_2|$$

In general, if $\vec{x}, \vec{y} \in \mathbb{R}^d$

$$\mathrm{dist}_M(\vec{x},\vec{y}) := |x_1-y_1| + |x_2-y_2| + \cdots + |x_d-y_d|$$

$$= \sum_{j=1}^{d} |x_j-y_j|$$

Def :

$$\|\vec{v}\|_M := \sum_{j=1}^{d} |v_j| .$$

Exercise : Check that $\|\cdot\|_M$ verifies all the properties fulfilled by the Eucl. norm, namely pos, vanishing, hom, triang ineq.

Limit of a function

(of vector var, vector valued)

We aim to def

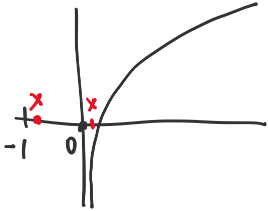
$$\lim_{\vec{x}\to \enclose{circle}{\vec{x}_0}} \vec{f}(\vec{x}) = \vec{\ell}$$

where $\vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$

and $\vec{x}_0 \in \mathbb{R}^d \cup \{\infty_d\}$, $\vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}$

First: to make sense, pt. $\vec{x}_0$ cannot be any point.

$f(x) = \log x$

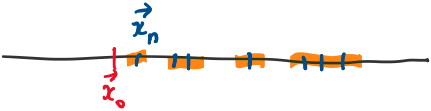


$$\lim_{x \to -1} \log x \text{ is not defd}$$

$$\lim_{x \to 0+} \log x$$

Def: We say that $\vec{x}_0$ is an accumulation pt for D if

$$\exists\, (\vec{x}_n) \subset D, \quad \vec{x}_n \longrightarrow \vec{x}_0, \quad \vec{x}_n \neq \vec{x}_0$$



Rmk: We need $\vec{x}_n \neq \vec{x}_0$ to prevent this case

$\vec{x}_0$ $\quad D = \{\vec{x}_0\}$

If the def of acc pt is

$$\vec{x}_0 \text{ acc pt} \Leftrightarrow \exists (\vec{x}_n) \subset D : \vec{x}_n \longrightarrow \vec{x}_0$$

then $\vec{x}_0$ would be a acc pt of $D=\{x_0\}$ but in this case there's nothing of D close to $\vec{x}_0$ apart for pt $\vec{x}_0$ itself.

If the def is

$$\vec{x}_0 \ \underline{\text{acc}} \text{ pt of } D \Leftrightarrow \exists (\vec{x}_n) \subset D : \underset{\underline{\vec{x}_n \neq \vec{x}_0}}{\vec{x}_n \longrightarrow \vec{x}_0}$$

then $\vec{x}_0$ is not acc. pt for $D=\{\vec{x}_0\}$.

Rmk: In prev def, pt $\vec{x}_0$ can be

$$\vec{x}_0 \in \mathbb{R}^d \quad \text{or} \quad \vec{x}_0 = \infty_d .$$

∞_d is acc pt for D $\Longleftrightarrow$

$$\exists (\vec{x}_n) \subset D^{\subset \mathbb{R}^d} : \vec{x}_n \longrightarrow \infty_d , \ \vec{x}_n \neq \infty_d$$

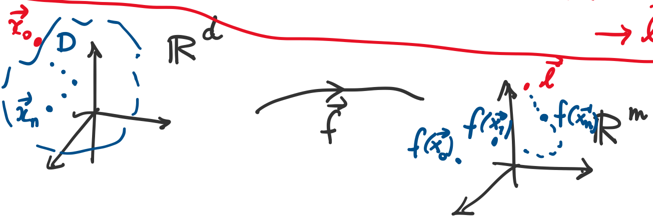
(here restriction $\vec{x}_n \neq \infty_d$ is redundant)

Def: Let $\vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x}_0$ be an acc. pt. for D. We say that

$$\lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}$$

if

$$\boxed{\forall (\vec{x}_n) \subset D,\ \vec{x}_n \longrightarrow \vec{x}_0,\ \vec{x}_n \neq \vec{x}_0 \Rightarrow \vec{f}(\vec{x}_n) \to \vec{\ell}}$$



Let's focus on the case $d = 2, 3$, $m = 1$

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) \qquad f : D \subset \mathbb{R}^2 \to \mathbb{R}$$

$$\lim_{(x,y,z)} f(x,y,z) \qquad f : D \subset \mathbb{R}^3 \to \mathbb{R}.$$

Example 1

$$\lim_{(x,y) \to (0,0)} \boxed{\frac{xy}{x^2+y^2}}$$

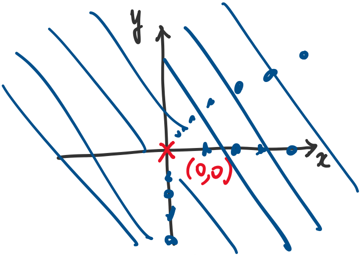
Here $f(x,y) = \dfrac{xy}{x^2+y^2}$, nat. defd on

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\}$$

Now $x^2+y^2 = 0 \iff x^2 = y^2 = 0 \iff$

$$\iff (x,y) = (0,0)$$

$$\Rightarrow D = \mathbb{R}^2 \setminus \{(0,0)\}$$



Since we have to compute $\lim\limits_{(x,y)\to(0,0)} f$

we need to check whether (0,0) is acc. pt for D or not.

$$\iff \exists \big((x_n,y_n)\big) \subset D : \underset{(0,0)}{\overset{}{(x_n,y_n) \to (0,0)}}$$

$$\qquad (x_n,y_n) \neq (0,0)$$

Clearly yes! $\Rightarrow$ limit makes sense.

When

$$(x,y) \to (0,0) \Leftrightarrow \begin{array}{l} x \to 0 \\ y \to 0 \end{array}$$

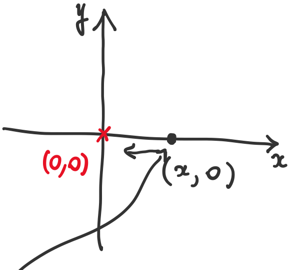
$$\Rightarrow \begin{array}{l} xy \to 0 \\ x^2+y^2 \to 0 \end{array}$$

$$f = 0/0 \quad \text{I.F.}$$

Pb: To compare

$N = xy$ and $D = x^2+y^2$

we must det how fast $N, D \to 0$



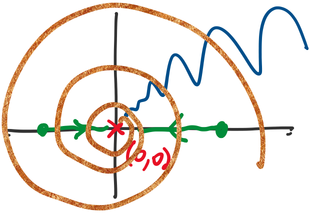
$$(x,0) \to (0,0) \Leftrightarrow x \to 0$$

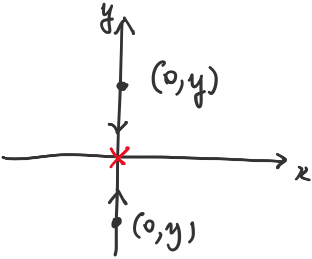
$$f(x,0) = \frac{x \cdot 0}{x^2 + 0^2} = \frac{\overset{N}{0}}{\underset{D}{x^2}} = 0$$

$$f(x,0) \equiv 0 \longrightarrow 0$$

$$\forall x \qquad \begin{array}{c} x \to 0 \\ \Updownarrow \\ (x,0) \to (0,0) \end{array}$$

$$? \Rightarrow \quad \lim_{\boxed{(x,y)\to(0,0)}} f(x,y) = 0 \quad ?$$





$$(0,y) \to (0,0) \Leftrightarrow y \to 0$$

$$f(0,y) = \frac{0\cdot y}{0^2 + y^2} = \frac{0}{y^2} \equiv 0 \xrightarrow[y\to 0]{} 0$$

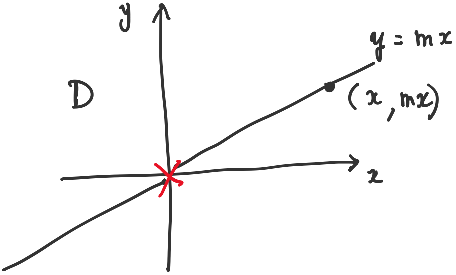
$$\Updownarrow$$

$$(0,y) \to (0,0)$$

$$\lim f(x,0) = 0$$

$\Rightarrow \qquad (x,0) \to (0,0)$

$$\lim_{(0,y)\to(0,0)} f(0,y) = 0$$



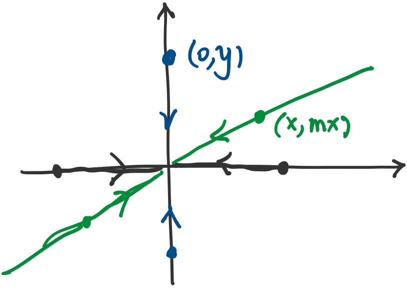
$$\underbrace{(x, mx) \longrightarrow (0,0)} \quad \Leftrightarrow \quad x \to 0$$

$$f(x, mx) = \frac{x \cdot mx}{x^2 + (mx)^2} = \frac{mx^2}{x^2 + m^2x^2}$$

$$= \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{m}{1+m^2}$$

$$\begin{tikzcd} \frac{m}{1+m^2} \arrow[d, "{\begin{array}{c} x \to 0 \\ \Updownarrow \\ (x, mx) \to (0,0) \end{array}}"] \\ \frac{m}{1+m^2} \end{tikzcd}$$

$$\Rightarrow \lim_{(x,mx)\to(0,0)} f(x,mx) = \frac{m}{1+m^2}.$$



$$f(x,0) \equiv 0 \to 0$$

$$f(0,y) \equiv 0 \to 0$$

$$f(x,mx) \equiv \frac{m}{1+m^2}$$

$$\to \frac{m}{1+m^2}$$

Conclusion: since it is not true that

$$\forall (\vec{x}_n) \subset D,\ \vec{x}_n \longrightarrow \vec{0},\ \vec{x}_n \neq \vec{0}$$

$$\Downarrow$$

$$f(\vec{x}_n) \to \ell$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y). \blacksquare$$

Rmk: This example shows an interesting test to disprove the existence of a limit:

if $\quad \exists\ (\vec{x}_n) \subset D,\ \vec{x}_n \to \vec{x}_0,\ \vec{x}_n \neq x_0$

$\quad \exists\ (\vec{y}_n) \subset D,\ \vec{y}_n \to \vec{x}_0,\ \vec{y}_n \neq x_0$

st.

$$\begin{aligned} \vec{f}(\vec{x}_n) &\longrightarrow \vec{l}_1 \\ \vec{f}(\vec{y}_n) &\longrightarrow \vec{l}_2 \end{aligned} \quad \neq$$

$$\Rightarrow \quad \nexists \lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x})$$

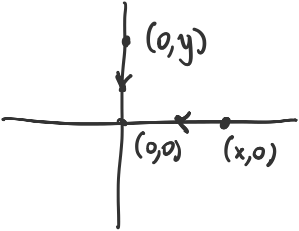
Do Ex 2.7.6.

# 3 $\displaystyle \lim_{(x,y) \to (0,0)} \frac{y^2 - xy}{x^2 + y^2}$.

Sol: Here $f(x,y) = \dfrac{y^2 - xy}{x^2+y^2}$,

defined on $D = \{(x,y) : x^2+y^2 \neq 0\}$

$= \mathbb{R}^2 \setminus \{(0,0)\}$.



$$\begin{tikzcd} f(x,0) = \frac{0}{x^2} \equiv 0 \arrow[d, "x \to 0"] \\ 0 \end{tikzcd}$$

$$x \to 0 \iff (x,0) \to (0,0)$$

$$f(0,y) = \frac{y^2}{y^2} \equiv 1 \xrightarrow{y \to 0} 1$$

$$\Updownarrow$$

$$(0,y) \to (0,0)$$

$$\Rightarrow \quad \nexists \lim_{(x,y)\to(0,0)} f. \qquad \blacksquare$$

Do ex 2.7.6 # 1,2,4,5,6.