Wednesday, 25 October 2023 08:43

Errata - corrige

Ex 1.5.16

Eqn $$y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}.$$

Goal : define

$$\lim_{\vec{x} \to \vec{x}_0} \vec{f}(\vec{x}) = \vec{\ell}.$$

Part case : $\lim_{n \to \infty} \vec{x}_n = \vec{\ell}$

Recall that

Def: Given a sequence $(x_n) \subset \mathbb{R}$, we say that

$$\exists \lim_{n \to +\infty} x_n = \ell \in \mathbb{R}$$

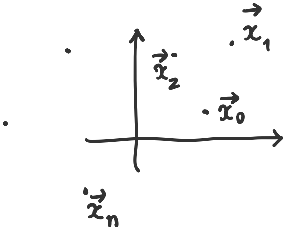
if

$$\boxed{\forall \varepsilon>0 \quad \exists N\in\mathbb{N} : |x_n-\ell| \leq \varepsilon , \quad \forall n \geq N}$$

$$\downarrow$$

$$\operatorname{dist}(x_n,\ell)$$

We may adapt this def to the case of $(\vec{x}_n) \subset \mathbb{R}^d$

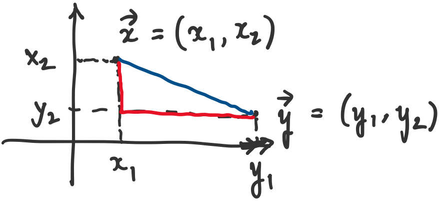


saying that $\exists \lim\limits_{n\to+\infty} \vec{x}_n = \vec{\ell}$ if

$$\forall \varepsilon>0 \quad \exists N\in\mathbb{N} : \boxed{\operatorname{dist}(\vec{x}_n,\vec{\ell})} \leq \varepsilon, \quad \forall n \geq N.$$

But how $\operatorname{dist}(\vec{x},\vec{y})$ can be defined?

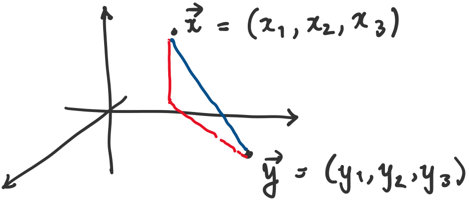
$d=2$



$$\operatorname{dist}(\vec{x},\vec{y}) = \sqrt{(y_1-x_1)^2+(x_2-y_2)^2}$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$d = 3$



$$\text{dist}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

$d = 1$ $\quad \vec{x} = (x_1)$

$\vec{y} = (y_1)$

$$\text{dist}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2}$$

$$= |x_1 - y_1|$$

Gen idea : if $\vec{x}, \vec{y} \in \mathbb{R}^d$

$\vec{x} = (x_1 \text{———} x_d)$

$\vec{y} = (y_1 \text{———} y_d)$

$$\text{dist}(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2}$$

$$= \sqrt{\sum_{j=1}^{d} (x_j - y_j)^2} .$$

Def (Euclidean norm)

If $\vec{v} \in \mathbb{R}^d$, $\vec{v} = (v_1, \dots, v_d)$

$$\underbrace{\|\vec{v}\|}_{\text{euclidean norm of } \vec{v}} := \sqrt{\sum_{j=1}^{d} v_j^2} .$$

Ex: $d = 2$

$$\|(1,1)\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$ ■

Rmk: So

$$\operatorname{dist}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$$

Def: Given $(\vec{x}_n) \subset \mathbb{R}^d$, we say that

$$\exists \lim_{n \to \infty} \vec{x}_n = \vec{\ell} \in \mathbb{R}^d$$

if

$$(*) \quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \overset{\delta_n \geq 0}{\underbrace{\|\vec{x}_n - \vec{\ell}\|}_{\text{positive real numbers}}} \leq \varepsilon, \ \forall n \geq N.$$

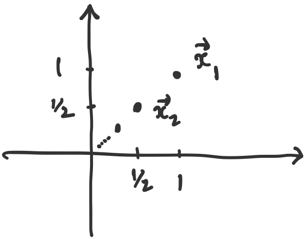
Rmk: If $\delta_n = \|\vec{x}_n - \vec{\ell}\|$,

$$\lim_{n\to\infty} \vec{x}_n = \vec{\ell} \iff \boxed{\forall \varepsilon>0\ \exists N : 0\leq \delta_n \leq \varepsilon,\ \forall n\geq N}$$

$$\iff \lim_{n\to\infty} \delta_n = 0$$

$$\lim_{n\to\infty} \vec{x}_n = \vec{\ell} \in \mathbb{R}^d \iff \lim_{n\to\infty} \|\vec{x}_n - \vec{\ell}\| = 0$$

Ex: • $\vec{x}_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ $n \geq 1$



n	$\vec{x}_n$
1	$(1,1)$
2	$\left(\frac{1}{2}, \frac{1}{2}\right)$
3	$\left(\frac{1}{3}, \frac{1}{3}\right)$

Guess: $\lim_{n\to\infty} \left(\frac{1}{n}, \frac{1}{n}\right) = \underset{\underset{\vec{\ell}}{\shortparallel}}{\vec{0}}$

To check this, we have to check that

$$\lim_{n\to\infty} \|\vec{x}_n - \vec{\ell}\| = 0$$

$$\Leftrightarrow \quad \lim_{n \to \infty} \underbrace{\left\| \left(\frac{1}{n}, \frac{1}{n} \right) - (0,0) \right\|}_{} = 0 \ ?$$

$$\left\| \left(1/n , 1/n \right) \right\| = \sqrt{\frac{1}{n^2} + \frac{1}{n^2}}$$

$$= \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n}$$

$$\Leftrightarrow \quad \lim_{n \to \infty} \frac{\sqrt{2}}{n} = 0 \ ?$$ yes! ■

- $$\lim_{n \to +\infty} \left(\boxed{\frac{1}{n}} , \boxed{\frac{n+1}{n}} \right) \ ? \quad \overset{?}{=} (0,1) \ ?$$

To check this let's compute

$$\lim_{n \to \infty} \left\| \vec{x}_n - \vec{\ell} \right\| = \lim_{n \to \infty} \left\| \left(\frac{1}{n}, \frac{n+1}{n} \right) - (0,1) \right\|$$

$$= \lim_{n \to \infty} \left\| \left(\frac{1}{n}, \underset{\shortparallel}{\frac{n+1}{n} - 1} \right) \right\|$$

$1/n$

$$= \lim_{n\to\infty} \left\|\left(\frac{1}{n}, \frac{1}{n}\right)\right\| = \lim_{n\to\infty} \frac{\sqrt{2}}{n} = 0$$

yes, $\lim_{n} \left(\frac{1}{n}, \frac{n+1}{n}\right) = (0,1)$. $\blacksquare$

Rmk: Is it possible to compute limits of sequences of vectors component wise?

Precisely, let $(\vec{x}_n) \subset \mathbb{R}^d$,

$$\vec{x}_n = (x_{1,n}, \; x_{2,n}, \; x_{3,n}, \dots, x_{d,n})$$

$$\left[\begin{array}{lcl} \vec{x}_1 & = & (x_{1,1}, \; x_{2,1}, \; \text{---} \; x_{d,1}) \\ \vec{x}_2 & = & (x_{1,2}, \; x_{2,2}, \; \text{- - -} \; x_{d,2}) \\ \vdots & & \\ \vec{x}_n & = & (x_{1,n}, \; x_{2,n}, \; \text{- - -} \; x_{d,n}) \\ \vdots & & \end{array}\right.$$

Guess:

$$\lim_{n\to\infty} \vec{x}_n = \vec{\ell} \in \mathbb{R}^d \quad \overset{?}{\boxed{\iff}}$$

$$\lim_{n\to\infty} x_{j,n} = \ell_j \qquad \forall j = 1, \dots d$$

This is true.

Proof:

$$\text{Hp}: \boxed{\lim_{n\to\infty} \vec{x}_n = \vec{\ell}} \iff \boxed{\lim \|\vec{x}_n - \vec{\ell}\| = 0}$$

$$\Rightarrow$$

$$\text{Th}: \boxed{\lim_{n\to\infty} x_{j,n} = \ell_j}$$

$$\|\vec{x}_n - \vec{\ell}\| = \sqrt{\sum_{j=1}^{d} (x_{j,n} - \ell_j)^2}$$

$$\geq \sqrt{(x_{j,n} - \ell_j)^2} = |x_{j,n} - \ell_j|$$

$$\Rightarrow \quad \underset{\swarrow}{\|\vec{x}_n - \vec{\ell}\|} \geq \underbrace{|x_{j,n} - \ell_j|}_{\downarrow} \geq 0 \quad \forall j = 1 \dots d$$

$$0 \quad \Rightarrow \quad 0$$ (police thm)

$\Rightarrow \quad x_{j,n} \to \ell_j \quad n \to +\infty, \quad \forall j = 1 \ldots d$

Th.

$\Leftarrow$ (do by exercise)

Hp: $\lim x_{j,n} = \ell_j \in \mathbb{R} \quad \forall j = 1 \ldots d$

Th: $\|\vec{x}_n - \vec{\ell}\| \to 0.$

■

Example:

$$\vec{x}_n = \left(1/n, \sin n\right)$$

Here

$$x_{1,n} = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

$$x_{2,n} = \sin n \not\to$$

$$\Rightarrow \nexists\, \vec{\ell} \in \mathbb{R}^2 : \quad \vec{x}_n \to \vec{\ell}.$$

■

Do 2.7.2 / 3

Some rmks on eucl. norm

Prop: Let $\|\cdot\|$ be the euclidean norm on $\mathbb{R}^d$. The following properties hold:

i) $\|\vec{v}\| \geq 0 \quad \forall \vec{v} \in \mathbb{R}^d$ (positivity)

ii) $\|\vec{v}\| = 0 \iff \vec{v} = \vec{0}$ (vanishing)

iii) $\|\lambda \vec{v}\| = |\lambda| \, \|\vec{v}\| \quad \lambda \in \mathbb{R}, \vec{v} \in \mathbb{R}^d$ (homogeneity)

iv) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \quad \forall v, w \in \mathbb{R}^d$ (triang ineq).

Proof: i) follows from def $\| \ \| = \sqrt{\ldots}$

ii) $\|\vec{v}\| = 0 \iff \vec{v} = (v_1 — v_d)$

$$\sqrt{\sum_{j=1}^{d} v_j^2} = 0 \qquad \sqrt{\bullet} = 0$$

$$\iff \sum_{j=1}^{d} v_j^2 = 0$$

$$\iff v_j^2 = 0 \quad \forall j = 1 — d$$

$$\iff v_j = 0 \qquad 〃$$

$$\iff \vec{v} = \vec{0}.$$

iii) $\|\lambda \vec{v}\| = \|\lambda (v_1 —— v_d)\|$

$$= \| (\lambda v_1, \lambda v_2, \ldots, \lambda v_d) \|$$

$$= \sqrt{\sum_{j=1}^{d} (\lambda v_j)^2}$$

$$\boxed{\lambda^2} v_j^2$$

$$= \sqrt{\lambda^2 \sum_{j=1}^{d} v_j^2} = |\lambda| \sqrt{\sum_{j=1}^{d} v_j^2}$$

$$= |\lambda| \, \|\vec{v}\| .$$

iv) Goal: $\boxed{\|\vec{v} + \vec{w}\|} \leq \|\vec{v}\| + \|\vec{w}\|$.

$$\vec{v} = (v_1 \text{ --- } v_d)$$

$$\vec{w} = (w_1 \text{ --- } w_d)$$

$$\vec{v} + \vec{w} = (v_1 + w_1, \text{ --- } v_d + w_d)$$

$$\|\vec{v} + \vec{w}\|^2 = \sum_{j=1}^{d} (v_j + w_j)^2$$

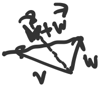
$$= \underbrace{\sum_{j=1}^{d} \Big(v_j^2 + w_j^2}_{} + 2 v_j w_j . \Big)$$

$$= \underbrace{\sum_{j=1}^{d} v_j^2}_{\|\vec{v}\|^2} + \underbrace{\sum_{j=1}^{d} w_j^2}_{\|\vec{w}\|^2} + 2\sum_{j=1}^{d} v_j w_j$$

$$\Rightarrow \quad \|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\underbrace{\sum_{j=1}^{d} v_j w_j}_{(\vec{v}\cdot\vec{w})}$$

$$\left(\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\boxed{\vec{v}\cdot\vec{w}} \right) \quad (*)$$

Lemma (Cauchy-Schwarz inequality)



$$|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\| \qquad \forall\, \vec{v},\vec{w} \in \mathbb{R}^d$$

Applying the C-S ineq. to (*)

$$\boxed{\vec{v}\cdot\vec{w}} \leq |\vec{v}\cdot\vec{w}| \leq \boxed{\|\vec{v}\|\,\|\vec{w}\|}$$

$$\Rightarrow \quad \|\vec{v}+\vec{w}\|^2 \leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\,\|\vec{w}\|$$

$$= \left(\|\vec{v}\| + \|\vec{w}\| \right)^2$$

$$\Rightarrow \qquad \|\vec v+\vec w\|^2 \leq \left(\|\vec v\|+\|\vec w\|\right)^2$$

$$\Rightarrow \qquad \|\vec v+\vec w\| \leq \|\vec v\|+\|\vec w\| \qquad \blacksquare$$

Proof of C-S inequality: we prove first

$$\vec v \cdot \vec w \leq \underbrace{\|\vec v\| \cdot \|\vec w\|}$$

If $\vec v=\vec 0$ or $\vec w=\vec 0$ ineq is trivial

If $\vec v \neq \vec 0$ and $\vec w \neq 0$ $\Rightarrow$ we have to prove

$$\frac{\boxed{\vec v \cdot \vec w}}{\|\vec v\|\,\|\vec w\|} \leq 1$$

$$\sum_{j=1}^{d} \frac{v_j\, w_j}{\|\vec v\|\,\|\vec w\|} \leq 1$$

$$\Updownarrow$$

$$\sum_{j=1}^{d} \overset{a}{\boxed{\frac{v_j}{\|\vec v\|}}}\ \overset{b}{\boxed{\frac{w_j}{\|\vec w\|}}} \leq 1$$

(elementary ineq : $\boxed{ab \leq \frac{1}{2}(a^2+b^2)}$

$$2ab \leq a^2+b^2 \iff a^2+b^2-2ab \geq 0$$

$$(a-b)^2 \geq 0$$

$$\sum_j \frac{v_j}{\|\vec{v}\|}\,\frac{w_j}{\|\vec{w}\|} \;\leq\; \sum_j \boxed{\frac{1}{2}}\left(\frac{v_j^2}{\boxed{\|\vec{v}\|^2}} + \frac{w_j^2}{\|\vec{w}\|^2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{\|\vec{v}\|^2}\overset{\|\vec{v}\|^2}{\overset{\shortparallel}{\sum_{j=1}^{d} v_j^2}} + \frac{1}{\|\vec{w}\|^2}\overset{\|\vec{w}\|^2}{\overset{\shortparallel}{\sum_j w_j^2}}\right]$$

$$\shortparallel$$

$$1 \quad + \quad 1$$

$$= 1. \qquad \blacksquare$$

Do 2.7.1.