The gen sol of

$$y'' + by' + cy = \underbrace{f(t)}$$

$(b, c \in \mathbb{R}, \; f(t) \in \mathcal{C}(I)$ known$)$

is

$$\boxed{y(t) = c_1 w_1(t) + c_2 w_2(t) + u(t)}$$

where

- (w_1, w_2) is a fund syst. of sols of $y'' + by' + cy = 0$
- u is a part sol of NH eqn given by the formula

$$u(t) = c_1(t)\, w_1(t) + c_2(t)\, w_2(t)$$

$$c_1(t) = -\int \frac{f(t)}{W(t)} w_2(t)\, dt$$

$$c_2(t) = \int \frac{f(t)}{W(t)} w_1(t)\, dt$$

where

$$W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}$$

Ex 1.5.10.

Gen int of

1. $y'' + y' - 6y = \boxed{2e^{-t}}$

Sol: Let's first compute the fund syst of sols for hom eqn

$$y'' + y' - 6y = 0$$

Ch eqn : $\lambda^2 + \lambda - 6 = 0$

$$\Delta = 1^2 - 4 \cdot 1 \cdot (-6) = +25 > 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

$$= 2, -3$$

$$\boxed{\begin{aligned} w_1 &= e^{2t} \\ w_2 &= e^{-3t} \end{aligned}}$$

Second step: let's compute a part sol of the NH eqn

$$u(t) = \underbrace{c_1(t)\, w_1(t)}_{} + c_2(t)\, w_2(t)$$

where

$$c_1(t) = -\int \frac{f(t)}{\underline{W(t)}}\, w_2(t)\, dt \qquad c_2(t) = \int \frac{f(t)}{W(t)}\, w_1(t)\, dt$$

where $W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}$

$$= \det \begin{bmatrix} e^{2t} & e^{-3t} \\ 2e^{2t} & -3e^{-3t} \end{bmatrix}$$

$$= -3e^{-t} - 2e^{-t} = \underbrace{-5e^{-t}}_{}$$

$$\Rightarrow \quad c_1(t) = -\int \frac{2\cancel{e^{-t}}}{-5\cancel{e^{-t}}} \cdot e^{-3t}\, dt$$

$$= +\frac{2}{5}\int e^{-3t}\, dt = \frac{2}{5}\, \frac{e^{-3t}}{-3}$$

$$= \boxed{-\frac{2}{15}\, e^{-3t}}$$

$$c_2(t) = +\int \frac{2\cancel{e^{-t}}}{-5\cancel{e^{-t}}} \cdot e^{2t}\, dt$$

$$= -\frac{\cancel{2}}{5}\int 2e^{2t}\, dt = \underbrace{-\frac{1}{5} e^{2t}}_{}$$

$$\Rightarrow u(t) = -\frac{2}{15} \underbrace{e^{-3t} \cdot e^{2t}}_{e^{-t}} - \frac{3}{15} \underbrace{e^{2t} \cdot e^{-3t}}_{e^{-t}}$$

$$= -\frac{1}{3} e^{-t}$$

Conclusion: the gen sol is

$$\boxed{y(t) = c_1 e^{2t} + c_2 e^{-3t} - \frac{1}{3} e^{-t} \qquad c_1, c_2 \in \mathbb{R}}$$

∎

1.5.11

Solve the CP

$$\begin{cases} y'' + y = t \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Sol: Step 1: gen sol of

$$y'' + y = \boxed{t}$$

$$y = c_1 w_1(t) + c_2 w_2(t) + u(t) \quad (NH)$$

where

- (w_1, w_2) is fund syst of sols of

$$y'' + y = 0$$

- u is a part sol of (NH)

Ch. eqn : $\cancel{\lambda^2 + \lambda = 0}$

$$\lambda^2 + 1 = 0 \qquad \Delta < 0$$

$$\lambda_{1,2} = \pm i = \overset{\alpha}{0} \pm i \, \overset{\beta}{1}$$

$$w_1 = \cos t \qquad w_2 = \sin t$$

To compute u, we apply the Lagrange formula

$$u(t) = \left(-\int \frac{f(t)}{W(t)} w_2(t)\,dt\right) w_1(t) + \left(\int \frac{f(t)}{W(t)} w_1(t)\,dt\right) w_2(t$$

$$= \left(-\int \frac{t}{1} \sin t \,dt\right) \cos t + \left(\int \frac{t}{1} \cos t \,dt\right) \sin t$$

$$W(t) = \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} = \det \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$= \cos^2 t + \sin^2 t \equiv 1$$

$$\int t \sin t \, dt = \int t \, (-\cos t)' = -t\cos t - \int 1 \cdot (-\cos t$$

$$= -t\cos t + \int \cos t \, dt$$

$$= -t\cos t + \sin t$$

$$\int t\cos t \, dt = \int t \cdot (\sin t)' \, dt = t\sin t + \int - \sin t \, dt$$

$$= t\sin t + \cos t$$

$$\Rightarrow u(t) = (t\cos t \underbrace{- \sin t) \cos t}_{} + (t\sin t + \underbrace{\cos t) \sin t}_{}$$

$$= t(\cos^2 t + \sin^2 t) = t$$

$\Rightarrow$ Concl: the gen sol is

$$y(t) = c_1 \cos t + c_2 \sin t + t, \quad c_1, c_2 \in \mathbb{R}$$

Step 2: y solves the CP $\Leftrightarrow$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 \cdot 0 + 0 = 0 \\ -c_1 \cdot 0 + c_2 \cdot 1 + 1 = 0 \end{cases}$$

$$y' = -c_1 \sin t + c_2 \cos t + 1$$

$$\Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = -1 \end{cases}$$

the sol of CP is $\boxed{y(t) = -\sin t + t}$

■

General Fact

Proposition The C.P

$$\begin{cases} y'' + by' + cy = f(t) \quad (\in \mathcal{C}(I)) \\ y(t_0) = y_0 \\ y'(t_0) = v_0 \end{cases}$$

has always $(\forall t_0 \in I, \ \forall y_0, v_0 \in \mathbb{R})$ a unique sol.

Proof: The gen sol of the eqn is

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + u(t)$$

(where (w_1, w_2) is fund syst of sols for
$y'' + by' + cy = 0$)

y solves the C.P. $\iff$

$$\begin{cases} y(t_0) = y_0 \\ y'(t_0) = v_0 \end{cases} \iff \begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) + u(t_0) = y_0 \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) + u'(t_0) = v_0 \end{cases}$$

$$y'(t) = c_1 w_1'(t) + c_2 w_2'(t) + u'(t)$$

$\uparrow$ system in c_1, c_2

$$\iff \begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) = y_0 - u(t_0) \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) = v_0 - u'(t_0) \end{cases}$$

$$\underbrace{\begin{bmatrix} w_1(t_0) & w_2(t_0) \\ w_1'(t_0) & w_2'(t_0) \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 - u(t_0) \\ v_0 - w'(t_0) \end{bmatrix}$$

(gen fact you checked as ex.)

$$\Longleftrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}^{-1} \begin{bmatrix} \downarrow \\ \quad \end{bmatrix}$$

■

Do Ex 1.5.11/$\underbrace{12/13/14/15/16}$.

Euclidean space $\mathbb{R}^d$

In the remaining part of course we will work on functions of vector variable, vector valued.

Examples:

$$f(\underbrace{x,y}) = xy \qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$f(x,y) = e^{x}y^{2} + x$$

$$\{(x,y) : x, y \in \mathbb{R}\} =: \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

$$f(x,y,z) = xyz \qquad f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

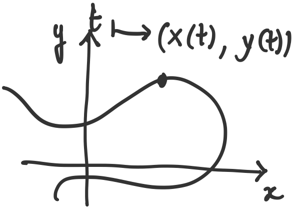
$$= xy^2 + ze^x$$

$$= z\sqrt{x^2+y^2-1} \qquad f : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\{(x,y,z) : x,y,z \in \mathbb{R}\} =: \mathbb{R}^3$$

$$f(x) = (x, x^2) \qquad \boxed{f : \mathbb{R} \longrightarrow \mathbb{R}^2}$$

$$f(1) = (1,1)$$
$$f(2) = (2,4)$$

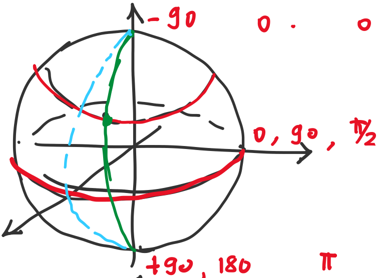


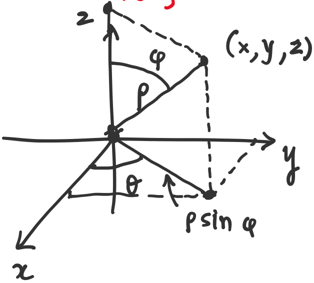
$$f\underbrace{(x,y)} = \overbrace{(x+y, xy)}$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$f(x,y) = (x+y, xy, x^2+y^2)$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$





$$\begin{cases} z = \rho\cos\varphi \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases}$$

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

If ρ is fixed (as on the surface of the Earth), $\rho = R$ radius of E.

$$\begin{cases} x = R \sin\varphi \cos\theta \\ y = R \sin\varphi \sin\theta \\ z = R \cos\varphi \end{cases}$$

$$\begin{array}{ccc} \mathbb{R}^2 & & \mathbb{R}^3 \\ (\varphi,\theta) & \longrightarrow & (x,y,z) \end{array}$$

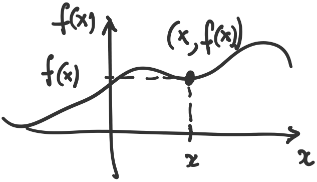
In general we define

$$\mathbb{R}^d := \Big\{ \overbrace{\underbrace{(x_1,\dots,x_d)}_{\text{vector}}}^{\vec{x}} \;:\; x_1,\dots,x_d \in \mathbb{R} \Big\}$$

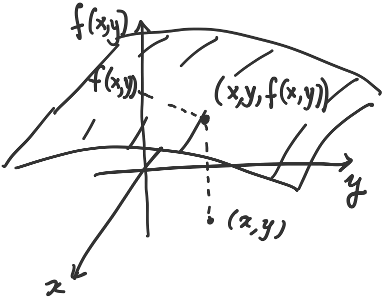
and we will consider functs.

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m .$$

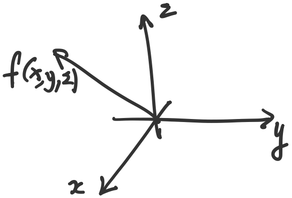
$$f = f(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$$



$$f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$



$$f = f(x,y,z) : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$$



$\mathbb{R}^d$ has a natural structure of vector space : on $\mathbb{R}^d$ is defd a sum

$$\underset{\vec{x}}{(x_1 \cdots x_d)} \underset{+}{\oplus} \underset{\vec{y}}{(y_1 \cdots y_d)} := (x_1 \oplus y_1, \dots, x_d + y_d)$$

and a product by a scalar (real numbers)

$$\underset{\in \mathbb{R}}{\lambda} \underset{\vec{x}}{(x_1 \ldots x_d)} := (\lambda x_1, \lambda x_2, \ldots, \lambda x_d)$$

These two operations verify few simple properties:

- (associativity of $+$)

$$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$$

- (commutativity)

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

- (zero) $\vec{0} = (0, 0, \ldots 0)$

$$\vec{x} + \vec{0} = \vec{x}$$

- (opposite)

$$\forall \vec{x} \; \exists \vec{y} : \quad \vec{x} + \vec{y} = \vec{0}$$

$$\vec{x} = (x_1 \text{——} x_d)$$

$$\vec{y} = (-x_1, \ldots, -x_d)$$

Prod:

- (assoc.)

$$\lambda(\mu\vec{x}) = (\lambda\mu)\vec{x} \qquad \lambda,\mu \in \mathbb{R}$$

$$\vec{x} \in \mathbb{R}^d$$

- (1)

$$1\vec{x} = \vec{x}$$

Distributivity

$$(\lambda+\mu)\vec{x} = \lambda\vec{x} + \mu\vec{x}$$

$$\lambda(\vec{x}+\vec{y}) = \lambda\vec{x} + \lambda\vec{y}.$$