Thursday, 19 October 2023 08:45

$\boxed{\Delta = 0}$ $\quad \lambda^2 + b\lambda + c = 0 \quad$ $\lambda_1 = \lambda_2 \in \mathbb{R}$

Since steps done for $\Delta > 0$ can be repeated up to formula

$$y(t) = \tilde{k}\, e^{\lambda_2 t} + k\, e^{\lambda_2 t} \int e^{(\lambda_1 - \lambda_2)t}\, dt$$

we have (for $\lambda_1 = \lambda_2$)

$$y(t) = \tilde{k}\, e^{\lambda_2 t} + k\, e^{\lambda_2 t} \overset{t + c}{\overset{\parallel}{\int 1\, dt}}$$

$$= \tilde{k} \boxed{e^{\lambda_2 t}} + k \boxed{t\, e^{\lambda_2 t}}$$

Conclusion: the gen sol of

$$y'' + by' + cy = 0$$

when $\Delta = 0$ (of ch. eqn $\lambda^2 + b\lambda + c = 0$) is

$$y(t) = k_1 e^{\lambda_1 t} + k_2\, t e^{\lambda_1 t}$$

$k_1, k_2 \in \mathbb{R}$

Rmk: As for case $\Delta = 0$, gen sol is linear combination of two particular sols:

$\Delta > 0$, $\quad w_1(t) = e^{\lambda_1 t} \quad w_2(t) = e^{\lambda_2 t}$

$\Delta = 0$, $\quad w_1(t) = e^{\lambda_1 t} \quad w_2(t) = t e^{\lambda_1 t}$

Ex: Gen int of

$$y'' - 2y' + y = 0$$

Sol: Ch eqn: $\lambda^2 - 2\lambda + 1 = 0 \iff (\lambda - 1)^2 = 0$

$\lambda_1 = \lambda_2 = 1 \quad (\Delta = 0)$.

Gen sol: $\quad y(t) = k_1 e^t + k_2 t e^t,$

$k_1, k_2 \in \mathbb{R}$ ■

Case $\Delta < 0$. Ch eqn $\lambda^2 + b\lambda + c = 0$

has roots $\lambda_1, \lambda_2 \in \mathbb{C}$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm i\sqrt{-\Delta}}{2a}$$

$$\sqrt{\Delta} = \sqrt{-(-\Delta)} = \sqrt{-1}\,\sqrt{-\Delta} = i\sqrt{-\Delta}$$

$$= -\underbrace{\frac{b}{2a}}_{\in \mathbb{R}} \pm i \underbrace{\frac{\sqrt{-\Delta}}{2a}}_{\in \mathbb{R}}$$

$$\boxed{\lambda_{1,2} = \alpha \pm i\beta} \qquad \alpha, \beta \in \mathbb{R}$$

$$\lambda_1 \neq \lambda_2$$

$$\left(e^{x+iy} = e^{x} \underbrace{e^{iy}} = e^{x}(\cos y + i \sin y)\right)$$

(Euler Formula $e^{iy} := \cos y + i \sin y$)

$2^{\sqrt{2}}$ $\qquad$ $y \in \mathbb{R}$

Borrowing the arg used for $\Delta > 0$ we can write that the gen int is

$$y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

$$e^{\lambda_1 t} = e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t}$$

$$= \boxed{e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)}$$

$$e^{\lambda_2 t} = e^{\alpha t}\left(\cos(-\beta t) + i\sin(-\beta t)\right)$$

$$= \boxed{e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)}$$

$$+$$

In part

$$y(t) = \frac{1}{2}\cdot e^{\lambda_1 t} + \frac{1}{2}\cdot e^{\lambda_2 t}$$ is a sol

$$= e^{\alpha t}\cos(\beta t) =: w_1(t)$$

$$y(t) = -\frac{i}{2}\, e^{\lambda_1 t} + \frac{i}{2}\, e^{\lambda_2 t}$$

$$= e^{\alpha t}\sin(\beta t) =: w_2(t)$$

In general, it can be proved that the gen int for $\Delta < 0$ is

$$y(t) = k_1 \underbrace{e^{\alpha t}\cos(\beta t)}_{w_1} + k_2 \underbrace{e^{\alpha t}\sin(\beta t)}_{w_2}$$

Ex: Gen int of

$$y'' + y = 0$$

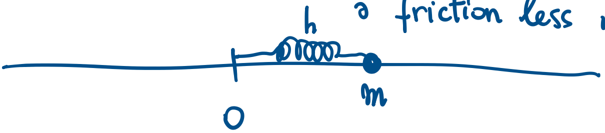
Sol: Ch. eqn : $\xcancel{\lambda^2 + \lambda} = 0$ $\qquad \lambda^2 + 1 = 0$

$$\Rightarrow \quad \lambda_{1,2} = \pm i = \underset{\alpha}{0} \pm i \underset{\beta}{1}$$

Gen sol : $y(t) = k_1 e^{0t} \cos(1t) + k_2 e^{0t} \sin(1 \cdot t)$

$$= k_1 \cos t + k_2 \sin t. \quad ∎$$

Ex: Mass m under elastic force on a friction less rail



1. Determine the poss motions
2. Find the sol st. $y(0) = \ell$, $y'(0) = v$.

Q1: The Newton eqn is

$$ma = F \iff m y''(t) = -h y(t)$$

$$\iff \boxed{\underset{\overset{\vee}{0}}{m} y'' + \underset{\overset{\vee}{0}}{h} y = 0}$$

Let's det the gen int of this eqn:

Ch. eqn : $m\lambda^2 + h = 0$

$$\Leftrightarrow \lambda^2 = -\frac{h}{m} < 0 \qquad (\Delta < 0)$$

$$\Leftrightarrow \lambda_{1,2} = \pm i\sqrt{\frac{h}{m}}.$$

$$= 0 \pm i\sqrt{\frac{h}{m}}$$

$$\Rightarrow y(t) = k_1 \cos\left(\sqrt{\frac{h}{m}}\, t\right) + k_2 \sin\left(\sqrt{\frac{h}{m}}\, t\right).$$

$$k_1, k_2 \in \mathbb{R}$$

Q2. To find the motion s.t. $y(0) = \ell$

$y'(0) = v$

we need to find k_1, k_2 s.t.

$$\begin{cases} y(0) = \ell \\ y'(0) = v \end{cases} \Leftrightarrow \begin{cases} k_1 + k_2 \cdot 0 = \ell \\ k_1 \cdot 0 + k_2\sqrt{\frac{h}{m}} \cdot 1 = v \end{cases}$$

$\nearrow$

$$y'(t) = -k_1\sqrt{\frac{h}{m}} \sin\left(\sqrt{\frac{h}{m}}\, t\right) + k_2\sqrt{\frac{h}{m}} \cos\left(\sqrt{\frac{h}{m}}\, t\right)$$

$$\Longleftrightarrow \quad \begin{cases} k_1 = \ell \\ \sqrt{\frac{h}{m}}\, k_2 = v \end{cases} \quad \Longleftrightarrow \quad \begin{cases} k_1 = \ell \\ k_2 = \sqrt{\frac{m}{\ell}} \cdot v \end{cases}$$

∎

The eqn

$$y'' + by' + cy = 0$$

has gen int

$$y(t) = k_1 w_1(t) + k_2 w_2(t), \quad k_1, k_2 \in \mathbb{R}$$

where w_1, w_2 are two part sols

- $w_1 = e^{\lambda_1 t}$, $w_2 = e^{\lambda_2 t}$ when $\Delta > 0$
  λ_1, λ_2 sols of ch. eqn
- $w_1 = e^{\lambda_1 t}$, $w_2 = t e^{\lambda_1 t}$ when $\Delta = 0$
  λ_1 sol of ch eqn
- $w_1 = e^{\alpha t} \cos(\beta t)$, $w_2 = e^{\alpha t} \sin(\beta t)$ when $\Delta < 0$
  $\lambda_{1,2} = \alpha \pm i\beta$ sols of ch. eqn.

(w_1, w_2) are called fund system of sols.

We now move to the non homog eqn

$$y'' + by' + cy = f(t)$$

($b, c \in \mathbb{R}$, $f \in \mathscr{C}(I)$ are known).

Prop: Suppose that $u = u(t)$ is a part sol of

$$\text{(NH)} \qquad y'' + by' + cy = f(t)$$

Then, the gen sol of (NH) is

$$y = \underbrace{k_1 w_1(t) + k_2 w_2(t)}_{\text{gen sol of hom eqn}} + \boxed{u(t)}.$$

Proof: If y solves (NH)

$$\boxed{y'' + by' + cy = f(t)} \quad -$$

and since also u solves (NH)

$$\boxed{u'' + bu' + cu = f(t)} \quad =$$

$$\underbrace{y'' - u''}_{} + b\underbrace{(y' - u')}_{} + c(y-u) = 0$$

$$(y-u)'' + b(y-u)' + c(y-u) = 0$$

$$\Rightarrow \quad y - u = k_1 w_1 + k_2 w_2 \qquad \blacksquare$$

Focus on $u(t)$ (part sol of (NH))

Idea: Lagrange trick

$$u(t) = k_1(t) w_1(t) + k_2(t) w_2(t)$$

where $k_1(t)$, $k_2(t)$ have to be found in such a way u solves (NH).

Let's write

$$\boxed{u = k_1 w_1 + k_2 w_2}$$

and let's impose

$$\enclose{circle}{u''} + bu' + cu = f.$$

$$u' = (k_1 w_1 + k_2 w_2)' = (k_1 w_1)' + (k_2 w_2)'$$

$$= \underbrace{k_1' w_1 + k_1 w_1'}_{(k_1 w_1)'} + \underbrace{k_2' w_2 + k_2 w_2'}_{(k_2 w_2)'}$$

We impose $\boxed{k_1' w_1 + k_2' w_2 \equiv 0}$

to simplify u' obtaining

$$\boxed{u' = k_1 w_1' + k_2 w_2'}$$

$$u'' = (k_1 w_1' + k_2 w_2')' = (k_1 w_1')' + (k_2 w_2')'$$

$$= \underbrace{k_1' w_1' + k_1 w_1''}_{(k_1 w_1')'} + \underbrace{k_2' w_2' + k_2 w_2''}_{(k_2 w_2')'}$$

So, u solves (NH) $\Leftrightarrow$

$$\underbrace{\boxed{k_1' w_1'} + \boxed{k_1 w_1''} + \boxed{k_2' w_2'} + \boxed{k_2 w_2''}}_{u''} + \boxed{b \, \underbrace{(k_1 w_1'}_{u'}} \underbrace{+ k_2 w_2')}_{}$$

$$+ \boxed{c \, (k_1 w_1} \underbrace{+ k_2 w_2)}_{u} = \boxed{f} \qquad (*)$$

$$k_1 \underbrace{\left(w_1'' + b w_1' + c w_1 \right)}_{= 0} = 0$$

$$k_2 \left(w_2'' + b w_2' + c w_2 \right) = 0 ,$$

$$\Leftrightarrow \quad \boxed{k_1' w_1' + k_2' w_2' = f}$$

Cond: $u = k_1 w_1 + k_2 w_2$ solves (NH) $\Leftrightarrow$

$$\begin{cases} k_1' w_1 + k_2' w_2 = 0 \\ k_1' w_1' + k_2' w_2' = f \end{cases}$$

$$\Leftrightarrow \quad \underbrace{\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}}_{\text{Wronskian matrix}} \begin{bmatrix} k_1' \\ k_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

It can be checked that

$$W := \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} \neq 0$$

in all the three cases ($\Delta > 0$, $\Delta = 0$, $\Delta \lneq 0$)

(do this check please)

$$\Rightarrow \quad \begin{bmatrix} k_1' \\ k_2' \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{cases} k_1'(t) = \dfrac{w_2}{W} f \\ \\ k_2'(t) = - \dfrac{w_1}{W} f \end{cases}$$

$$\Rightarrow \quad \begin{cases} k_1 = \displaystyle\int \frac{w_2}{W} f \, dt \\ \\ k_2 = - \displaystyle\int \frac{w_1}{W} f \, dt \end{cases}$$

Do Ex 1.5.9, 1.5.10, 1.5.11.