Monday, 16 October 2023 08:34

Ex 1.5.6

Solve the C.P.

2.

$$\begin{cases} y' = \dfrac{y(2y-1)}{\operatorname{Ch} t} & t \in \mathbb{R} \\ \\ y(0) = 1 \end{cases}$$

Sol: We have a sep vars eqn

$$y' = \underbrace{\frac{1}{\operatorname{Ch} t}}_{a(t)} \cdot \underbrace{y(2y-1)}_{f(y)}$$

$a \in \mathcal{C}$, $f \in \mathcal{C}^1$. Then, if $y(t)$ is a sol

- either $y(t) \equiv c \Leftrightarrow 0 = \underset{\neq 0}{\underbrace{\frac{1}{\operatorname{Ch} t}}} \cdot \underbrace{c(2c-1)}$

$$\Leftrightarrow c(2c-1) = 0$$

$$\Leftrightarrow \boxed{C = 0 \vee C = \tfrac{1}{2}.}$$

- or $y(t)$ is non const $\Rightarrow f(y(t)) \neq 0$

$$y' = \frac{1}{\text{Ch}\, t}\, y(2y-1) \Leftrightarrow \overbrace{\frac{y'}{y(2y-1)}}^{A} = \overbrace{\frac{1}{\text{Ch}\, t}}^{B}$$

$$\Leftrightarrow (*) \underbrace{\int \frac{y'(t)}{y(t)(2y(t)-1)}\, dt}_{} = \underbrace{\int \frac{1}{\text{Ch}\, t}\, dt \; + C}_{}$$

$$\left(A(t) = B(t) \Leftrightarrow \int A(t)\, dt = \int B(t)\, dt + c \right)$$

$$\int \frac{1}{\text{Ch}\, t}\, dt = \int \frac{1}{\frac{e^{t} + e^{-t}}{2}}\, dt$$

$$= 2 \int \frac{1}{e^{t} + \frac{1}{e^{t}}}\, dt$$

$$= 2 \int \frac{e^{t}}{1 + \underset{\overset{\shortparallel}{(e^{t})^{2}}}{e^{2t}}}\, dt$$

$$(\operatorname{arctg} e^t)' = \frac{1}{1+(e^t)^2} \cdot e^t$$

$$= 2 \operatorname{arctg}(e^t).$$

$$\int \frac{y'(t)}{y(t)(2y(t)-1)}\, dt \underset{\substack{u = y(t) \\ du = y'(t)\, dt}}{=} \boxed{\int \frac{1\, du}{u(2u-1)}}$$

$$= \int -\left(\frac{1}{u} + \frac{-2}{2u-1}\right) du$$

$$= -\log|u| + \log|2u-1|$$

$$= \log\left|\frac{2u-1}{u}\right| = \log\left|2 - \frac{1}{u}\right|$$

$$= \log\left|2 - \frac{1}{y(t)}\right|$$

So

$$(*) \iff \boxed{\log\left|2 - \frac{1}{y(t)}\right| = 2\,\text{arctg}(e^t) + c \qquad c \in \mathbb{R}}$$

⌈

$$\left|2 - \frac{1}{y(t)}\right| = e^{2\,\text{arctg}\, e^t + c}$$

$$= k\, e^{2\,\text{arctg}(e^t)} \qquad k > 0$$

$$2 - \frac{1}{y(t)} = \pm k\, e^{2\,\text{arctg}(e^t)}$$

$$2 - \frac{1}{y(t)} \equiv k\, e^{2\,\text{arctg}(e^t)} \qquad k \neq 0$$

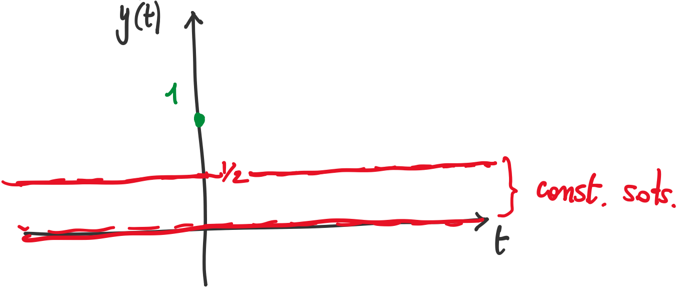
$$\iff \frac{1}{y(t)} = 2 - k\, e^{2\,\text{arctg}(e^t)}$$

$$\iff y(t) = \frac{1}{2 - k\, e^{2\,\text{arctg}\, e^t}} \,. \qquad k \neq 0$$

Conclusion: the gen int of the eqn is

$y \equiv 0, \frac{1}{2}$ ↗ $y = \dfrac{1}{2 - k\, e^{2\,\text{arctg}(e^t)}}$ $k \neq 0$

To solve the C.P. we have to det. which of prev. sols verifies the pass. cond $y(0) = 1$



Certainly $y \equiv 0$, $y \equiv 1/2$ are not sols of CP. A non const sol verifies $y(0) = 1$

$$\Leftrightarrow \underbrace{\frac{1}{2 - k\, e^{2 \underset{= \pi/4}{\operatorname{arctg} 1}}}}_{y(0)} = 1$$

$$\Leftrightarrow \frac{1}{2 - k e^{\pi/2}} = 1$$

$$\Leftrightarrow \quad 2 - k e^{\pi/2} = 1$$

$$\Leftrightarrow \quad k e^{\pi/2} = 1$$

$$\Leftrightarrow \quad k = e^{-\pi/2} \qquad \blacksquare$$

Ex 1.5.8

CP $\begin{cases} y' = y(1-y^2) \\ y(0) = 1/2 \end{cases}$

Q1 Det implicit form of sol. and, if poss., det the explicit form of y

Q2 Is it true that y is def'd $\forall\, t \in \mathbb{R}$?

Q1 Let's det the gen int of the eqn.

⌜ $$y' = \underset{a(t)y + b(t)}{y - y^3} \qquad y = e^{\int a(t)\,dt}\left[k + \int \dots\right]$$

No! ⌟

We have a sep vars eqn

$$y' = \underset{\substack{a(t)\\ \uparrow \\ \mathcal{C}}}{1} \cdot \underbrace{y(1-y^2)}_{\substack{f(y)\\ \in \mathcal{C}^1}}$$

Sol of this eqn are

- either $y \boxed{\equiv} C \Leftrightarrow 0 = C(1-C^2)$

$$\Leftrightarrow C = 0, \pm 1$$

there're three const sols

$y \equiv 0, \quad y \equiv 1, \quad y \equiv -1.$

- or y is not const and $\underset{\neq 0}{f(y(t))}$

so

$$y' = \overbrace{y(1-y^2)}^{\neq 0} \Leftrightarrow \frac{y'}{y(1-y^2)} = 1$$

$$\Leftrightarrow \int \frac{y'(t)}{y(t)(1-y(t)^2)}\,dt = \int 1\,dt + c = t + c$$

$$\| \quad u = y(t)$$

$$du = y'(t)\,dt$$

$$\int \frac{du}{u(1-u^2)} = \int \frac{1-u^2+u^2}{u(1-u^2)}\,du$$

$$= \int \frac{1}{u}\,du + \frac{1}{-2}\int \frac{-2u}{1-u^2}\,du$$

$$= \log|u| - \frac{1}{2}\log|1-u^2|$$

$$= \log|y(t)| - \cancel{\frac{1}{2}}\log\left|1-y(t)^2\right|^{1/2}$$

$$= \log \frac{|y(t)|}{\sqrt{|1-y(t)^2|}}$$

So

$$\boxed{\log \frac{|y(t)|}{\sqrt{|1-y(t)^2|}} = t + c, \quad c \in \mathbb{R}}$$

implicit form of sol
for y non const

$$\Updownarrow$$

$$\frac{|y(t)|}{\sqrt{|1-y(t)^2|}} = e^{t+c} = k e^{t} \quad k>0$$

$$\left|\frac{y(t)^2}{1-y(t)^2}\right| = k e^{2t} \quad k>0$$

$$\frac{y(t)^2}{1-y(t)^2} = \pm k e^{2k} = k e^{2t} \quad k \neq 0$$

$$\Leftrightarrow \quad y(t)^2 = k e^{2t}\left(1-y(t)^2\right)$$

$$\Leftrightarrow \quad y(t)^2\left(1+ke^{2t}\right) = k e^{2t}$$

$$\Leftrightarrow \quad y(t)^2 = \frac{k e^{2t}}{1+k e^{2t}}$$

$$\Leftrightarrow \quad y(t) = \pm\sqrt{\frac{k e^{2t}}{1+k e^{2t}}} \quad k \neq 0.$$

Conclusion: the gen sol of the eqn is

$$y(t) \equiv 0, \pm 1 \quad \text{or} \quad y(t) = \boxed{\pm}\sqrt{\frac{ke^{2t}}{1+ke^{2t}}} \quad k \in \mathbb{R}, \; k \neq 0$$

Q2 The sol of CP is:

$(y(0) = 1/2)$

$$y(t) = \pm\sqrt{\frac{ke^{2t}}{1+ke^{2t}}} \qquad k \neq 0$$

$$y(0) = 1/2 \iff \frac{1}{2} = \pm\sqrt{\frac{k}{1+k}}$$

$$\iff 1/2 = +\sqrt{\frac{k}{1+k}}$$

$$\iff 1/4 = \frac{k}{1+k}$$

$$\iff 1+k = 4k$$

$$\iff 3k = 1 \iff k = 1/3$$

$$\Rightarrow \quad y(t) = + \sqrt{\frac{\frac{1}{3} e^{2t}}{1 + \frac{1}{3} e^{2t}}} \; .$$

$\pm$ WRONG !

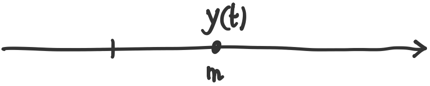
Clearly $y(t)$ is defd for every $t \in \mathbb{R}$. $\blacksquare$

Rmk: Solve the CP with $y(0) = 2$ and det for which $t \in \mathbb{R}$. $y(t)$ is defd. $\blacksquare$

Second Order Lin Eqns

Rmk: Second Newton Law

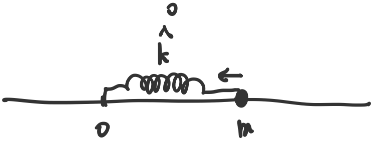
$$\boxed{m \vec{a} = \vec{F}}$$



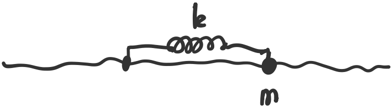
$$a = y''(t)$$

$$F = F(t, y(t), y'(t))$$

$$ma = F \iff m y''(t) = F(t, y(t), y'(t))$$



$$\boxed{m y''(t) = -k y(t)}$$



$$m y''(t) = -k y(t) - \overset{\overset{0}{\wedge}}{\nu} y'(t)$$

$$\boxed{a y''(t) + b y'(t) + c y(t) = 0}$$

(Lin homog 2nd order eqn with const. coeffs.)

$$\boxed{a y''(t) + b y'(t) + c y(t) = f(t)}$$

Lin non homog 2nd ord eqn with const coeffs.

Let's focus on the homog eqn

$$ay''(t) + by'(t) + cy(t) = 0$$

$a,b,c \in \mathbb{R}$ $\quad a \neq 0 \quad \Leftrightarrow$

$$(*) \quad \boxed{y''(t) + by'(t) + cy(t) = 0}$$

D = derivative $\qquad Dy = y'$

$\qquad\qquad\qquad\qquad\quad D^2 y = y''$

$$(*) \quad D^2 y + bDy + cy = 0$$

$$\Leftrightarrow \underbrace{(D^2 + bD + c)} y = 0$$

Characteristic polynomial :

$$\lambda^2 + b\lambda + c \qquad \lambda \in \cancel{\mathbb{R}}\ \mathbb{C}$$

Characteristic eqn

$$\boxed{\lambda^2 + b\lambda + c = 0}$$

Ex:

$$y'' + 3y' + 2y = 0 \rightsquigarrow \lambda^2 + 3\lambda + 2\cancel{\lambda} = 0$$

$$y'' - y' = 0 \rightsquigarrow \lambda^2 - \lambda = 0$$

$$y'' + y = 0 \rightsquigarrow \lambda^2 + 1 = 0$$

The ch. eqn can present three alternatives according to

$$\Delta = b^2 - 4c > 0, \quad = 0, \quad < 0$$

Case $\Delta > 0$

There're two real and different sols λ_1, λ_2:

$$\lambda^2 + b\lambda + c = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

$$D^2 + bD + c \overset{?}{=} (D - \lambda_1)(D - \lambda_2)$$

$$(D - \lambda_1)y = \boxed{y' - \lambda_1 y}$$

$$(D - \lambda_1)\underbrace{(D - \lambda_2)y}_{y' - \lambda_2 y} = (D - \lambda_1)\boxed{(y' - \lambda_2 y)}$$

$$= (y' - \underbrace{\lambda_2 y}_{})' - \lambda_1 (y' - \lambda_2 y)$$

$$= y'' - \lambda_2 y' - \lambda_1 y' + \lambda_1 \lambda_2 y$$

$$= y'' + (-\lambda_1 - \lambda_2) y' + \lambda_1 \lambda_2 y$$

$$\overset{?}{=} y'' + b y' + c y \quad \text{yes!}$$

So,

y is sol of $y'' + by' + cy = 0$

$$\Leftrightarrow \quad (D - \lambda_1)\boxed{(D - \lambda_2) y} = 0$$

To solve this eqn we may say that,

if $\boxed{u = (D - \lambda_2) y}$

$$\Rightarrow \quad (D - \lambda_1) u = 0$$

$$\Leftrightarrow \quad u' - \lambda_1 u = 0$$

$$\Leftrightarrow \quad u' = \lambda_1 u$$

$$\Leftrightarrow \quad u(t) = k e^{\lambda_1 t} \qquad k \in \mathbb{R}$$

$$\Rightarrow \quad (D-\lambda_2)\,y = u \iff (D-\lambda_2)\,y = k e^{\lambda_1 t}$$

$$\iff \boxed{y' - \lambda_2 y = k e^{\lambda_1 t}}$$

$$\iff \boxed{y' = \overset{a(t)}{\lambda_2} y + \overset{b(t)}{k e^{\lambda_1 t}}}$$

$$\iff y(t) = e^{\lambda_2 t}\left[\tilde{k} + \underbrace{\int e^{-\lambda_2 t} k e^{\lambda_1 t}\, dt}\right]$$

$$= \tilde{k} e^{\lambda_2 t} + k e^{\lambda_2 t} \int e^{(\lambda_1-\lambda_2)t}\, dt$$

$$y(t) = \tilde{k} e^{\lambda_2 t} + k e^{\lambda_2 t} \int e^{(\lambda_1-\lambda_2)t}\, dt$$

$$\overset{\lambda_1 \neq \lambda_2}{=} \tilde{k} e^{\lambda_2 t} + k e^{\lambda_2 t} \cdot \frac{e^{(\lambda_1-\lambda_2)t}}{\lambda_1-\lambda_2}$$

$$\boxed{y(t) = \tilde{k} e^{\lambda_2 t} + k e^{\lambda_1 t}}$$