Thursday, 12 October 2023 08:46

**Example:** Det sols of

$$y' = \underbrace{1+y^2}_{f(y)} \cdot \;=\; \underset{\overset{\equiv}{1}}{a(t)}\,\underset{(1+y^2)}{f(y)}$$

⌜

$$y' \overset{\curvearrowleft}{=} 1 + y^2 \iff \frac{y'}{y^2} = \underset{?}{\frac{1}{y^2}} + 1$$

⌟

$$y' = 1 + y^2 \iff \underset{\overset{\parallel}{\left(F(y(t))\right)'}}{\frac{y'}{1+y^2}} = 1$$

$$\boxed{\frac{y'(t)}{1+y(t)^2} = \overset{\parallel}{F'}(y(t))\, y'(t)}$$

$$\Uparrow$$

$$F'(y(t)) = \frac{1}{1+y(t)^2}$$

$$F'(y) = \frac{1}{1+y^2} \Rightarrow F(y) = \operatorname{arctg} y$$

$$\left(\operatorname{arctg} y(t)\right)' = \frac{1}{1+y(t)^2} \cdot y'(t)$$

$$\frac{y'}{1+y^2} = 1 \iff \left(\operatorname{arctg} y(t)\right)' = 1$$

$$\iff \operatorname{arctg} y(t) = \int 1\,dt + c$$

$$\boxed{\operatorname{arctg} y(t) = t+c}$$

$$\iff \boxed{y(t) = \operatorname{tg}(t+c)}$$

■

**Ex 2.**

$$y' = \underbrace{e^y}_{} \iff \frac{y'}{e^y} = 1$$

$$\iff e^{-y}\, y' = 1$$

$$\left(-e^{-y}\right)' = 1$$

$$\left(-e^{-y}\right)' = -e^{-y}\cdot(-y)' = +e^{-y}y'$$

$$\iff -e^{-y} = \int 1\,dt + c = t+c$$

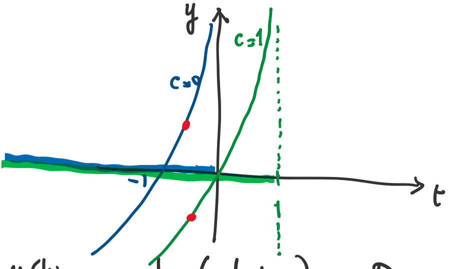
$$\iff \boxed{-e^{-y} = t+c}$$

$$\iff e^{-y} = -t+c$$

$$\iff -y = \log(-t+c)$$

$\Rightarrow \quad y = -\log(-t+c).$ ■

**Rmk:** $y' = 1 \cdot e^{y}$ $\quad a(t) \equiv 1 \quad t \in \mathbb{R}$



$$y(t) = -\log(\underbrace{-t+c}) \qquad D_y = ]-\infty, c[$$

$$-t+c>0 \Leftrightarrow t<c$$

$c=0 \qquad y = -\log(-t)$

$c=1 \qquad y = -\log(-t+1)$

■

In these few examples we considered eqns of type

$$y'(t) = a(t)\, f(y(t))$$

$$[\; y' = a(t) f(y) \;]$$

called sep. vars. eqn

**Rmk:**

$$y' = \frac{ty - t}{yt^2} = \frac{t(y-1)}{t^2 y} = \underbrace{\frac{1}{t}}_{a(t)} \cdot \underbrace{\frac{y-1}{y}}_{f(y)}$$

this is a sep vars eqn

$$y' = \frac{1}{t+y}$$ this is not a sep vars eqn

$$y' = y + t$$ " " " "

Catenary eqn :

$$(*) \quad \tau_0 \boxed{\alpha''(x)} - \rho g \sqrt{1 + \alpha'(x)^2} = 0$$

$$\begin{cases} y(x) := \alpha'(x) \leftarrow \\ \cancel{\tau_0}\, y'(x) = \dfrac{\rho g}{\tau_0} \sqrt{1 + y(x)^2} \end{cases} \iff y' = \underbrace{\boxed{\overset{\lambda}{\dfrac{\rho g}{\tau_0}}}}_{a(x)} \underbrace{\sqrt{1+y^2}}_{f(y)}$$

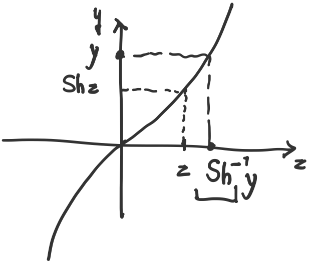
Let's solve $]0, +\infty[$

$$\downarrow$$

$$\boxed{y' = \underbrace{\lambda}_{a(x)} \underbrace{\sqrt{1+y^2}}_{f(y) \neq 0}} \iff \boxed{\frac{y'}{\sqrt{1+y^2}}} = \lambda \quad (**)$$

⌜

$$y = \operatorname{Sh} z = \frac{e^{z} - e^{-z}}{2}$$



$$\left(\operatorname{Sh}^{-1} y\right)' = \frac{1}{\sqrt{1+y^2}}$$

⌟

$$\frac{y'}{\sqrt{1+y^2}} = \left(\operatorname{Sh}^{-1} y(x)\right)'$$

$$(**) \iff \left(\operatorname{Sh}^{-1} y(x)\right)' = \lambda$$

$$\iff \boxed{\operatorname{Sh}^{-1} y(x) = \lambda x + c}$$

$$\iff \boxed{y(x) = \operatorname{Sh}(\lambda x + c)}$$

Going back to the cat. eqn, its sol is

$$\alpha'(x) = \operatorname{Sh}(\lambda x + c)$$

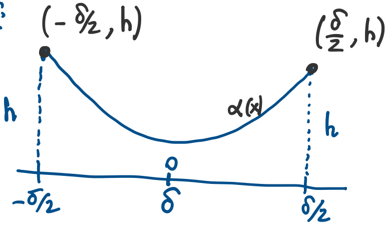
$$\alpha(x) = \underbrace{\int Sh(\lambda x + c)\, dx}_{c,\, d \in \mathbb{R}} + d$$

$$Sh = Ch'$$

$$Sh(\lambda x + c) = \left(\frac{1}{\lambda} Ch(\lambda x + c)\right)'$$

$$\boxed{\alpha(x) = \frac{1}{\lambda} Ch(\lambda x + \underset{\uparrow}{c}) + \underset{\nearrow}{d}, \quad c, d \in \mathbb{R}.}$$

**Example:**



h, δ

known data

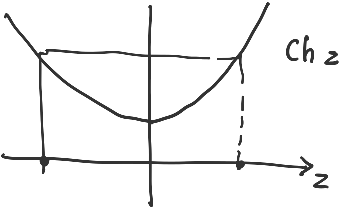
Our sol is the $\alpha(x)$ s.t.

$$\begin{cases} \alpha\left(-\frac{\delta}{2}\right) = h \iff \overbrace{\frac{1}{\lambda} Ch\left(-\frac{\delta}{2}\lambda + c\right) + d}^{\alpha(-\delta/2)} = h \\ \alpha\left(\frac{\delta}{2}\right) = h \iff \underbrace{\frac{1}{\lambda} Ch\left(\frac{\delta}{2}\lambda + c\right) + d = h} \end{cases}$$

$$\alpha(\delta/2)$$

$$\Leftrightarrow \begin{cases} \cancel{\frac{1}{\lambda}}\left(\mathrm{ch}\left(-\frac{\delta}{2}\lambda+c\right)-\mathrm{ch}\left(\frac{\delta}{2}\lambda+c\right)\right)=0 \\ d = h - \frac{1}{\lambda}\,\mathrm{ch}\left(\frac{\delta}{2}\lambda+c\right) \end{cases}$$

$$\mathrm{ch}\left(-\frac{\delta}{2}\lambda+c\right)=\mathrm{ch}\left(\frac{\delta}{2}\lambda+c\right)$$



$$\Leftrightarrow -\frac{\delta}{2}\lambda+\cancel{c}=\frac{\delta}{2}\lambda+\cancel{c} \quad \vee \quad -\frac{\delta}{2}\lambda+c=\ominus\left(\frac{\delta}{2}\lambda+c\right)$$

imposs

$$\Leftrightarrow \cancel{2}\,\boxed{c=0}$$

$$\Rightarrow \begin{cases} c=0 \\ d = h - \frac{1}{\lambda}\,\mathrm{ch}\left(\frac{\delta}{2}\lambda\right) \end{cases}$$

$$\Rightarrow \alpha(x)=\frac{1}{\lambda}\,\mathrm{ch}(\lambda x)+h-\frac{1}{\lambda}\,\mathrm{ch}\left(\frac{\delta}{2}\lambda\right)$$

In general, to solve a sep vars eqn

$$y' = a(t) f(y)$$

the key idea is to write

$$y' = a(t)\overset{f(y(t))}{f(y)} \overset{?}{\iff} \frac{y'}{f(y)} = a(t)$$

If $f(y) \neq 0$ $\forall y$ $\iff$ is true

But what if $f(y) = 0$ for some $y$?

If $\boxed{f(y_0) = 0}$ $\iff$ $y(t) \equiv y_0$ is a sol of eqn

**Prop:** $a \in \mathcal{C}(I)$, $f \in \mathcal{C}$, $a \not\equiv 0$.

$y(t) \equiv y_0$ is a sol of $y' = a(t) f(y)$ (*) $\iff$ $f(y_0) = 0$.

**Proof:** $\Rightarrow$ Hyp: $y(t) = y_0$ is a sol of $(*)$

$$\Downarrow$$

$$0 = a(t)\, f(y_0)$$

$$\Downarrow$$

$$f(y_0) = 0$$

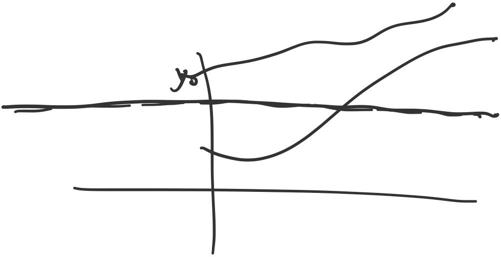
$\Leftarrow$ Hyp: $f(y_0) = 0$

Then $y(t) \equiv y_0 \Rightarrow y'(t) \equiv 0$

$$a(t) f(y(t)) = a(t) \underset{\overset{\|}{0}}{f(y_0)}$$

$$\equiv 0$$

$\Rightarrow$ $y(t)$ solves the eqn. ∎



$f(y_0) = 0$

**Thm:** Let $a \in \mathcal{C}$, $f \in \mathcal{C}^1$ $(f, f' \in \mathcal{C})$

Then sols of

$$y' = a(t) f(y)$$

are

- either constant (stationary) $y(t) \equiv y_0$ where $f(y_0) = 0$.
- or non constant and $f(y(t)) \neq 0$. In this case sol can be det. by sep of vars

$$y' = a(t) f(y) \iff \frac{y'}{f(y)} = a(t)$$

To solve $\dfrac{y'}{f(y)} = a(t)$

$$\|$$

step 1 : $\dfrac{y'}{f(y)} = \underline{(F(y))'} = F'(y)\, y'$

$$\iff F'(y) = \frac{1}{f(y)}$$

$$\iff F(y) = \int \frac{1}{f(y)}\, dy + c$$

step 2 : $F(y(t)) = \int a(t)\, dt + c$

step 3 : $\boxed{y(t) = F^{-1}\left(\int a(t)\, dt + c\right)}$

Do Ex 1.5.6 / 7 / 8.