Wednesday, 11 October 2023 08:40

Ex 1.5.1

Find the gen int of

1. $$y' + (\cos t)y = \frac{1}{2}\overset{2\sin t\cos t}{\sin(2t)} \qquad t \in \mathbb{R}$$

Sol: We have a first order NH eqn

$$y' = \underbrace{(-\cos t)}_{a(t)}y + \underbrace{\sin t\cos t}_{b(t)}$$

The gen int is

$$y(t) = e^{\int a(t)\,dt}\left[k + \int e^{-\int a(t)dt}\, b(t)\,dt\right]$$

$$= e^{\int -\cos t\,dt}\left[k + \int e^{\int \cos t\,dt}\,\sin t\cos t\,dt\right]$$

$$\int \cos t\,dt = \sin t$$

$$= e^{-\sin t}\left[k + \int \boxed{e^{\sin t}}\,\sin t\,\boxed{\cos t}\,dt\right]$$

$$e^{\sin t}\cos t$$

$$= \left(e^{\sin t}\right)'$$

$$\cdot \left(e^{\sin t} \cdot \overset{\cos t}{\overset{\|}{(\sin t)'}}\right)$$

$$\int e^{\sin t} \sin t \cos t \, dt = \int \left(e^{\sin t}\right)' \sin t \, dt$$

$$= e^{\sin t} \sin t - \int e^{\sin t} \underset{\cos t}{(\sin t)'} \, dt$$

$$= e^{\sin t} \sin t - e^{\sin t}$$

$$\Rightarrow \quad y(t) = e^{-\sin t}\left[k + e^{\sin t}(\sin t - 1)\right]$$

$$= k e^{-\sin t} + \sin t - 1, \quad k \in \mathbb{R}.$$

■

6. $y' + 2ty = t e^{-t^2} \quad t \in \mathbb{R}$

Sol: Here we have the eqn

$$y' = \boxed{-2t}\, y + \boxed{t e^{-t^2}}$$

$$\left[\; y' = \overset{a(t)}{a(t)}\, y + \overset{b(t)}{b(t)}, \qquad y' + a(t)y + b(t) = 0 \;\right]$$

The gen int is

$$y(t) = e^{-\int \cancel{+}2t\,dt}\left[k + \int e^{\int 2t\,dt} \cdot t e^{-t^2}\,dt\right]$$

$$= e^{-t^2}\left[k + \int \cancel{e^{t^2}} \cdot t\, \cancel{e^{-t^2}}\,dt\right]$$

$$= e^{-t^2}\left[k + t^2/2\right]. \qquad \blacksquare$$

1.5.3.

Eqn $\qquad y' - (\operatorname{tg} t)\, y = \dfrac{1}{\sin t} \qquad t \in \left]0, \pi/2\right[$

Q1 Gen int

Q2 $\lim\limits_{t \to 0+} y(t) = -\infty$ $\quad \forall\, y$ sol of Eqn?

Q3 Are there sols y : $\lim\limits_{t \to \pi/2 -} y(t) \in \mathbb{R}$

Sol

Q1 We have a NH lin eqn

$$y' = \underbrace{(\operatorname{tg} t)}_{a(t)} y + \underbrace{\frac{1}{\sin t}}_{b(t)}$$

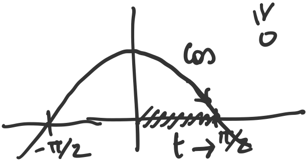
Gen sol:

$$y(t) = e^{\int \operatorname{tg} t \, dt} \left[k + \int e^{-\int \operatorname{tg} t \, dt} \cdot \frac{1}{\sin t} dt \right]$$

$$\int \operatorname{tg} t \, dt = -\int \frac{-\sin t}{\cos t} dt = -\log|\cos t|$$

$$\geq 0$$

$(t \in]0, \pi/2[$

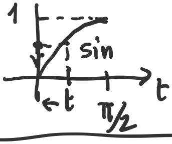


$$= -\log(\cos t)$$

$$y(t) = e^{\cancel{-} \log (\cos t)^{-1}} \left[k + \int e^{\log(\cos t)} \cdot \frac{1}{\sin t} dt \right]$$

$$= \frac{1}{\cos t} \left[k + \int \frac{\boxed{\cos t}}{\sin t} dt \right]$$

$$= \frac{1}{\cos t}\left[k + \log|\underset{\overset{\vee}{0}}{\sin t}|\right]$$



$$\boxed{y(t) = \frac{k + \log(\sin t)}{\cos t}} \qquad \begin{array}{l} k \in \mathbb{R} \\ t \in \left]0, \pi/2\right[\end{array}$$

Q2, We have to check if

$$\lim_{t \to 0+} \frac{\boxed{k} + \overset{-\infty}{\log}(\overset{0+}{\sin t})}{\underset{\downarrow\, 1}{\cos t}} \xrightarrow{} -\infty \quad = -\infty \quad \forall k \in \mathbb{R}$$

$\Rightarrow$ yes!

Q3. It asks

$$\exists k \quad : \quad \lim_{t \to \pi/2-} \frac{k + \log(\sin t)}{\cos t} \in \mathbb{R}\,?$$

$$\lim_{t\to\pi/2^-}\frac{\overset{\nearrow k}{k}+\log(\overset{\nearrow 1^-}{\sin t})\ \nearrow 0}{\underset{\swarrow\ 0^+}{\cos t}} \longrightarrow \underset{\frac{k}{0^+}}{k}$$

$$\frac{k}{0^+}=\begin{cases}+\infty & k>0\\ -\infty & k<0\end{cases}$$

$k=0 \qquad \frac{0}{0}$ I.F.

$$k=0 \qquad \lim_{t\to\pi/2^-}\frac{\log(\sin t)}{\cos t}=\frac{0}{0} \text{ I.F.}$$

$$\overset{H}{=}\lim_{t\to\pi/2^-}\frac{(\log(\sin t))'}{(\cos t)'}$$

$$=\lim_{t\to\pi/2^-}\frac{\frac{1}{\sin t}\cdot\cos t}{-\sin t}$$

$$= - \lim_{t \to \pi/2^-} \frac{\cos t \nearrow 0}{\underset{\swarrow 1 \quad \swarrow 1}{(\sin t)^2}} = 0$$

Conclusion : only for $k = 0$ $\lim y(t) \in \mathbb{R}$

$\Rightarrow$ there's just one sol st. $\lim\limits_{t \to \pi/2^-} y(t) \in \mathbb{R}$

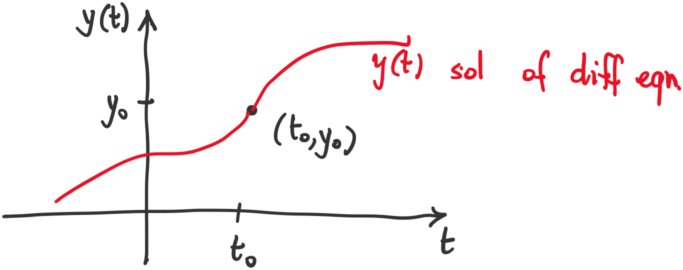
$$\left(y(t) = \frac{\log(\sin t)}{\cos t} \right). \qquad ■$$

Do 1.5.4.

Cauchy Problem

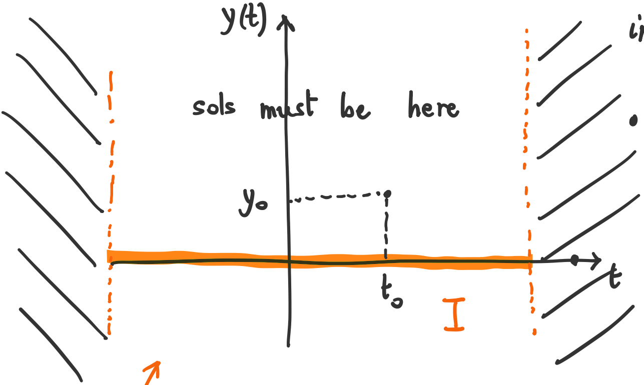
$$\begin{cases} y'(t) = \underbrace{a(t)}_{} \, y(t) + \underbrace{b(t)}_{} \\ y(t_0) = y_0 \end{cases}$$

where $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}$ is given.



Precisely: let $a = a(t)$, $b = b(t)$ are dfd for $t \in I \subset \mathbb{R}$

↑ interval



↗

$I \times \mathbb{R} = \{(t,y) : t \in I,\ y \in \mathbb{R}\}$

Thm: If $a, b \in \mathcal{C}(I)$, then

$\forall (t_0, y_0) \in I \times \mathbb{R} \quad \exists!$ sol $y = y(t)$, $t \in I$

(there exists a unique

of $CP(t_0, y_0)$:

$$\begin{cases} \boxed{y'(t) = a(t)\, y(t) + b(t)} \\ y(t_0) = y_0 \end{cases}$$

Proof: Recall that sols of

$$y' = a(t)\, y + b(t)$$

can be written as

$$\boxed{y(t) = k\, w_1(t) + u(t), \qquad k \in \mathbb{R}}$$

where $w_1 = e^{\overline{\qquad}} > 0$

A sol y verifies the pass. cond

$$\iff y(t_0) = y_0 \iff \boxed{k\, \underset{\neq 0}{w_1(t_0)} + u(t_0) = y_0}$$

$$\iff k = \frac{y_0 - u(t_0)}{w_1(t_0)} \qquad \blacksquare$$

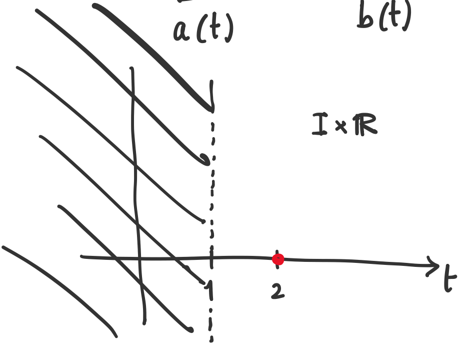
(unique k $\iff$ unique sol)

Ex 1.2.6

Solve $\begin{cases} y' - \dfrac{2y}{1-t^2} = t, & \boxed{\underline{t>1}} \\ y(2) = 0 \end{cases}$

Sol: The eqn is

$$y' = \underbrace{\frac{2}{1-t^2}}_{a(t)} y + \underbrace{t}_{b(t)}$$



$I =]1, +\infty[$

To find the sol of CP we start solving

the eqn : the gen sol is

$$y(t) = \underbrace{e^{\int \frac{2}{1-t^2}\,dt} \Big[k}_{k\ w_1} + \overbrace{\int e^{-\int \frac{2}{1-t^2}\,dt}}^{u} t\,dt \Big]$$

$$\int \frac{2}{1-t^2}\,dt = \int \frac{2}{(1-t)(1+t)}\,dt$$

$$\frac{2}{(1-t)(1+t)} = \frac{1}{1-t} + \frac{1}{1+t}$$

$$= \int \overset{= -\frac{1}{t-1}}{\frac{1}{1-t}} + \boxed{\frac{1}{1+t}}\,dt$$

$$= \log|t+1| - \log|t-1|$$

$$\overset{t>1}{=} \log\frac{t+1}{t-1}.$$

$$\Rightarrow \quad y(t) = e^{\log\frac{t+1}{t-1}} \left[k + \int e^{\cancel{-}\log\left(\frac{t+1}{t-1}\right)^{-1}} \cdot t\,dt \right]$$

$$= \frac{t+1}{t-1}\left[k + \int \frac{t-1}{t+1}\,\overset{(t+1-1)}{\cancel{t}}\,dt\right]$$

$$\int \frac{t-1}{\cancel{t+1}}\cancel{(t+1)} \ominus \frac{\overset{+1-1}{t}-1}{t+1}\,dt.$$

$$= \frac{t+1}{t-1}\left[k + \frac{1}{2}(t-1)^2 - \int \frac{\overset{1}{\cancel{t+1}}}{\cancel{t+1}} - \frac{2}{t+1}\,dt\right]$$

$$\boxed{y(t) = \frac{t+1}{t-1}\left[k + \frac{1}{2}(t-1)^2 - t + 2\log(t+1)\right]}$$

Second step: we impose $y(2)=0$

$$y(2)=0 \iff \cancel{3}\left[k + \frac{1}{2}1 - 2 + 2\log 3\right] = 0$$

$$\iff \boxed{k = \frac{3}{2} - \log 9.}$$

■

First order sep. vars. eqns

Recall the logistic eqn:

$$\frac{P'(t)}{P(t)} = -r\,(P(t) - M)$$

$y = P$
$\Leftrightarrow$

$$y' = -ry(y-M)$$
$$= +rMy - ry^2$$

⌜ A general form for these eqns could be

$$y' = f(y)$$

$$\boxed{y' = a(t)\,f(y)}$$
⌟

Ex 1: $y' = \underline{a(t)}\,y$ (first order lin homog eqn)

" $\Updownarrow$ "

$$(\log|y|)' = \frac{y'}{y} = a(t)$$

$$\Leftrightarrow \quad \log|y| = \int a(t)\,dt + c.$$

$\underline{\text{Ex 2}}$: $y' = 1 \cdot y^2$

$$\text{"}\Updownarrow\text{"}$$

$$(\qquad)' = \boxed{\frac{y'}{y^2}} = 1 \iff \left(-\frac{1}{y}\right)' = 1$$

$$\Updownarrow$$

$$(-y^{-1})' = +y^{-2}\,y'$$

$$\begin{aligned} -\frac{1}{y} &= \int 1\,dt + c \\ &= t + c \end{aligned}$$

$$\Rightarrow \quad -\frac{1}{y} = t + c$$

$$\Rightarrow \quad y = -\frac{1}{t + c} \qquad \blacksquare$$

Try to solve :

$$y' = 1 + y^2, \qquad y' = e^y, \qquad y' = y^3.$$