Monday, 9 October 2023 08:42

First Order Linear Eqns

Malthus eqn : $\boxed{\frac{P'(t)}{P(t)} = r}$ (const)

$$" \Updownarrow "$$

$$P'(t) = r\,P(t)$$

We consider now a eqn of type

$$\boxed{y'(t) = a(t)\cdot y(t)}$$

(first order lin. homog. eqn)

where a(t) is given.

Example: $y'(t) = \underset{\underset{a(t)}{\shortparallel}}{\boxed{t}}\, y(t)$

$$y'(t) = \underbrace{(\sin t)}_{a(t)}\, y(t)$$

Rmk: Why do we say "lin eqn"?

Because if y_1, y_2 are sols of the eqn

$$y_1'(t) = a(t)\, y_1(t)$$

$$y_2'(t) = a(t)\, y_2(t)$$

$$\Rightarrow \quad \boxed{\alpha y_1(t) + \beta y_2(t)} \qquad \forall \alpha, \beta \in \mathbb{R}$$

is a sol of the diff eqn.

Indeed:

$$\Big(\overbrace{\alpha y_1(t) + \beta y_2(t)}^{y'}\Big)' =$$

$$= \alpha \underset{a(t)y_1(t)}{\underset{\parallel}{y_1'}} + \beta \underset{a(t)y_2(t)}{\underset{\parallel}{y_2'}}$$

$$= a(t)\Big(\underbrace{\alpha y_1(t) + \beta y_2(t)}_{y}\Big) \qquad \blacksquare$$

How can we solve an eqn of type

$$y'(t) = a(t)\, y(t).$$

Idea: $y'(t) = a(t)\, y(t)$ " $\Longleftrightarrow$ " $\dfrac{y'(t)}{y(t)} = a(t)$

Precisely: if $y = y(t)$ is a sol s.t $y(t) \neq 0$ of

$$y'(t) = a(t)\,y(t) \iff \underset{\overset{\|}{\left(\log|y(t)|\right)'}}{\boxed{\frac{y'(t)}{y(t)}}} = a(t)$$

$$\iff \left(\underbrace{\log|y(t)|}_{\text{is one of the } (\infty \text{ many}) \text{ primitives of}}\right)' = a(t)$$

$$\iff \boxed{\log|y(t)| = \int a(t)\,dt + c, \qquad c \in \mathbb{R}}$$

(implicit form of sol)

$$\iff |y(t)| = e^{\int a(t)\,dt + c}$$

$$= e^{c}\, e^{\int a(t)\,dt}$$

$$|y(t)| = k\, e^{\int a(t)\,dt}, \qquad k>0$$

$$\Longleftrightarrow \qquad y(t) = \pm k\, e^{\int a(t)\,dt} \qquad k>0$$

$$\Longleftrightarrow \qquad \boxed{y(t) = k\, e^{\int a(t)\,dt} \qquad k\neq 0}$$

Since $y(t) \equiv 0$ is also a sol

Thm: The sols of

$$y'(t) = a(t)\,y(t)$$

gen sol/ gen integral

are given by the formula

$$\boxed{y(t) = k\, e^{\int a(t)\,dt}, \qquad k\in\mathbb{R}}$$

Rmk: There're infinitely many sols.

Rmk: $y'(t) = f(t)$ $\quad f(t)$ given

$$\Longleftrightarrow \qquad y(t) = \int f(t)\,dt + c$$

∎

Ex: Compute the gen int of

$$y'(t) = \boxed{t}\, y(t). \qquad (y' = ty)$$

Sol: We have a first ord. lin. homog eqn

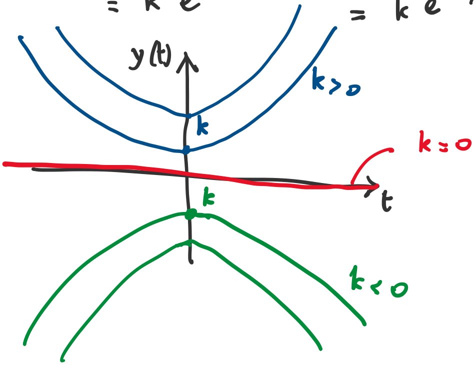
$$y' = a(t)y$$

where $a(t) = t$. The gen int is

$$y(t) = k\, e^{\int a(t)\,dt}$$

$$= k\, e^{\int t\,dt} = k\, e^{t^2/2} . \quad k\in\mathbb{R}.$$

■



Ex 2. Gen int of $y' = (\log t)\, y$ ■

$t \in]0, +\infty[$

Sol: We have an eqn of type

$y' = a(t)\,y$ where $a(t) = \log t$

Gen int is

$$y(t) = k\, e^{\int a(t)\,dt}$$

$$= k\, e^{\int \log t\,dt}$$

$$\int \log t\,dt = \int \underset{\overset{\shortparallel}{(t)'}}{1} \cdot \underline{\log t}\,dt = t\log t - \int t \cdot \frac{1}{t}\,dt$$

$$\int f g' = fg - \int f' g$$

$$= t\log t - \int 1\,dt$$

$$= t\log t - t.$$

$$\Rightarrow \quad y(t) = k\, e^{t\log t - t} \quad \blacksquare$$

$$k \in \mathbb{R}$$

Rmk: If $y'(t) = a(t)y(t)$ and a makes sense for $t \in I \subset \mathbb{R}$, every sol cannot be defined for $t \notin I$.

Ex: $y' = \underset{a(t)}{(\log t)}\,y \qquad I = \,]0, +\infty[$

every sol will be defined in an int.

c I. Taking $k=0$ we get

$$y \equiv 0$$ is a sol.

Also this function is defd only for $t>0$. □

" $$(\log t)' = \frac{1}{t}$$

Rmk: The gen int of

$$y' = a(t)\, y$$

is

$$y(t) = k \underbrace{e^{\int a(t)\,dt}}_{w_1(t)}$$

$$\boxed{y(t) = k\, w_1(t)} \quad k \in \mathbb{R}$$

First order non-homog eqns

We consider now eqns of type :

$$y'(t) = a(t)\,y(t) + b(t)$$

where $a = a(t)$, $b = b(t)$ are known both defd on an interval $I \subset \mathbb{R}$.

Prop: Suppose $u = u(t)$ be a part. sol of

$$y' = a(t)y + b(t) \qquad (NH)$$

Then the gen int of (NH) is

$$\boxed{y(t) = k w_1(t) + u(t), \quad k \in \mathbb{R}}$$

Proof: If y is a sol of (NH)

$$\boxed{y' = a(t)\,y + b(t).} \quad -$$

Since u is a sol of (NH)

$$\boxed{u' = a(t)u + b(t)} \quad =$$

$$\Rightarrow \quad y' - u' = a(t)(y-u)$$

$$\underbrace{(y-u)}{}' = a(t)(y-u)$$

$$\boxed{Y' = a(t)\,Y} \Rightarrow Y \text{ solves the hom eqn}$$

$$\Rightarrow \underset{y-u}{\underset{\shortparallel}{Y}} = k w_1(t)$$

$$\Rightarrow y = k w_1(t) + u, \quad k \in \mathbb{R}. \qquad \blacksquare$$

Goal: det. a part sol of

$$y' = a(t)y + b(t). \quad (NH)$$

Lagrange trick: $u(t) = \underset{\uparrow}{k(t)}\, w_1(t)$

to be found in such a way u is sol of (NH)

Let's plug this u into (NH):

$$u' = a(t)u + b(t)$$

$$u' = (k(t)w_1(t))' = \boxed{k'(t)w_1(t) + k(t)w_1'(t)}$$

$$\underbrace{k'(t)w_1(t) + \cancel{k(t)w_1'(t)}}_{u'(t)} = \cancel{a(t)\underbrace{k(t)w_1(t)}_{u(t)}} + b(t)$$

$$w_1'(t) = a(t)w_1(t) \qquad (w_1 \text{ solves hom. eqn})$$

$$\Leftrightarrow \qquad k'(t)w_1(t) = b(t)$$

$$\Leftrightarrow \qquad k'(t)\underbrace{e^{\int a(t)dt}}_{\neq 0} = b(t)$$

$$\Leftrightarrow \qquad \boxed{k'(t) = e^{-\int a(t)}\, b(t)}$$

$$\Leftrightarrow \qquad k(t) = \int \left(e^{-\int a(t)\,dt} b(t) \right) dt$$

$$\Rightarrow \qquad u(t) = \underbrace{\left(\int e^{-\int a(t)\,dt} b(t)\,dt \right) w_1(t).}_{}$$

Conclusion: The gen int of

$$y' = a(t)y + b(t) \qquad (NH)$$

is

$$y(t) = k\,w_1(t) + u(t)$$

$$= w_1(t) \left(k + \int e^{-\int a(t)\,dt} b(t)\,dt \right)$$

$$\boxed{y(t) = e^{\int a(t)\,dt} \left[k + \int e^{-\int a(t)\,dt} b(t)\,dt \right] \qquad k \in \mathbb{R}.}$$

Ex 1.2.4

Compute the gen int of

$$y'(t) - \frac{2}{t}\, y(t) = 1 \qquad t \in \,]0,+\infty[.$$

Sol: We have a first order non homog eqn

$$y' = \underbrace{\boxed{\frac{2}{t}}}_{a(t)} y + \underbrace{\boxed{1}}_{b(t)}$$

Gen int is

$$y(t) = e^{\int a(t)\,dt}\left[k + \int e^{-\int a(t)\,dt}\, b(t)\,dt\right]$$

$$= e^{2\int \frac{\cancel{2}^{1}}{t}\,dt}\left[k + \int e^{-\int \frac{2}{t}\,dt} \cdot 1\,dt\right]$$

$$\overset{t>0}{=} e^{\cancel{2}\log t^{2}}\left[k + \int e^{\cancel{-2}\log t^{-2}}\,dt\right]$$

$$= t^{2}\left[k + \underbrace{\int \frac{1}{t^{2}}\,dt}_{-1/t}\right]$$

$$= t^{2}\left[k - 1/t\right] = \boxed{kt^{2} - t}, \quad k \in \mathbb{R}. \quad \blacksquare$$

Do 1.5.1/2/3.